Robust Game-Theoretic Algorithms for Distributed Resource Allocation in Wireless Communications

by

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A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy by Loughborough University

CERTIFICATE OF ORIGINALITY

This is to certify that I am responsible for the work submitted in this thesis, that the original work is my own except as specified in acknowledgements or in footnotes, and that neither the thesis nor the original work contained therein has been submitted to this or any other institution for a degree.

.... (Signed)

Amod Jai Ganesh Anandkumar (candidate)

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1. The citations are listed individually in order to attach hyperlinks to the corresponding bibliography entries
2. The pages are renumbered in order to correspond to PDF page numbers

Abstract

The predominant game-theoretic solutions for distributed rate-maximization algorithms in Gaussian interference channels through optimal power control require perfect channel knowledge, which is not possible in practice due to various reasons, such as estimation errors, feedback quantization and latency between channel estimation and signal transmission. This thesis therefore aims at addressing this issue through the design and analysis of robust game-theoretic algorithms for rate-maximization in Gaussian interference channels in the presence of bounded channel uncertainty.

A robust rate-maximization game is formulated for the single-antenna frequency-selective Gaussian interference channel under bounded channel uncertainty. The robust-optimization equilibrium solution for this game is independent of the probability distribution of the channel uncertainty. The existence and uniqueness of the equilibrium are studied and sufficient conditions for the uniqueness of the equilibrium are provided. Distributed algorithms to compute the equilibrium solution are presented and shown to have guaranteed asymptotic convergence when the game has a unique equilibrium.

The sum-rate and the price of anarchy at the equilibrium of this game are analyzed for the two-user scenario and shown to improve with increase in channel uncertainty under certain conditions. These results indicate that the robust solution moves closer to a frequency division multiple access (FDMA) solution when uncertainty increases. This leads to a higher sum-rate and a lower price of anarchy for systems where FDMA is globally optimal.

A robust rate-maximization game for multi-antenna Gaussian interference channels in the presence of channel uncertainty is also developed along similar principles. It is shown that this robust game is equivalent to the nominal game with modified channel matrices. The robust-optimization equilibrium for this game and a distributed algorithm for its computation are presented and characterized. Sufficient conditions for the uniqueness of the equilibrium and asymptotic convergence of the algorithm are presented.

Numerical simulations are used to confirm the behaviour of these algorithms. The analytical and numerical results of this thesis indicate that channel uncertainty is not necessarily detrimental, but can indeed result in improvement of performance of networks in particular situations, where the Nash equilibrium solution is quite inefficient and channel uncertainty leads to reduced greediness of users.

I dedicate this thesis to my early mentors:

My grandfather, Y. K. Nanjundaiah
S. R. Madhu Rao
M. N. Krishna Swamy
K. Ramamurthy
V. Kesavan

Contents Overview

Statement of Originality

The following aspects of this thesis are believed to be original:

- The extension of the MIMO iterative waterfilling algorithm to broadband Gaussian interference channels and the study of the effect of channel estimation errors on the performance of the MIMO iterative algorithm in Chapter 3.
- The robust rate-maximization game formulation for frequency-selective SISO Gaussian interference channels and the analysis of its equilibrium (Theorem 4.2 on page 105) in Chapter 4.
- The analysis of the effect of channel uncertainty on the sum-rate in the robust SISO rate-maximization game for the two-user case (Theorem 5.1 on page 125 and Theorem 5.2 on page 128) in Chapter 5.
- The robust rate-maximization game formulation for MIMO Gaussian interference channels in Chapter 6.

The novelty of this thesis is supported by the following works:

1. Amod J.G. Anandkumar, Animashree Anandkumar, Sangarapillai Lambotharan, and Jonathon Chambers, "Robust Rate-Maximization Game for MIMO Gaussian Interference Channels Under Bounded Channel Uncertainty", *submitted to IEEE Transactions on Wireless Communications.*

2. Amod J.G. Anandkumar, Animashree Anandkumar, Sangarapillai Lambotharan, and Jonathon Chambers, "Robust Rate-Maximization Game Under Bounded Channel Uncertainty", *submitted to IEEE Transactions on Vehicular Technology.*

3. Amod J.G. Anandkumar, Animashree Anandkumar, Sangarapillai Lambotharan, and Jonathon Chambers, "Efficiency of Rate-Maximization Game Under Bounded Channel Uncertainty", *44th Asilomar Conference on Signals, Systems and Computers*, Pacific Grove, CA, Nov. 7 - 10 2010.

4. Amod J.G. Anandkumar, Animashree Anandkumar, Sangarapillai Lambotharan, and Jonathon Chambers, "Robust Rate-Maximization Game Under Bounded Channel Uncertainty", *35th IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2010)*, Dallas, TX, Mar. 14 - 19 2010.

5 Amod J.G. Anandkumar, Sangarapillai Lambotharan, and Jonathon Chambers, "Application of Distributed MIMO Waterfilling Algorithm for Broadband Channels and its Performance in the Presence of CSI Errors", *First UK-India International Workshop on Cognitive Wireless Systems (UKIWCWS 2009)*, New Delhi, India, Dec. 11-12 2009

6 Amod J.G. Anandkumar, Sangarapillai Lambotharan, and Jonathon Chambers, "A Game-Theoretic Approach to Transmitter Covariance Matrix Design for Broadband MIMO Gaussian Interference Channels," *IEEE/SP 15th Workshop on Statistical Signal Processing, 2009 (SSP '09)*, Cardiff, UK, Aug. 31 - Sept. 3 2009.

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Detailed Contents

Abbreviations

CSI	Channel state information
DSL	Digital subscriber line
FCC	Federal Communications Commission
FDMA	Frequency division multiple access
GIC	Gaussian interference channel
IWFA	Iterative waterfilling algorithm
KKT	Karush–Kuhn–Tucker
MIMO	Multiple-input multiple-output
MUI	Multi-user interference
NP	Non-deterministic polynomial time
OFDM	Orthogonal frequency-division multiplexing
QoS	Quality-of-service
RF	Radio frequency
SISO	Single-input single-output
s. t.	subject to
WiMAX	Worldwide Interoperability for Microwave Access

Notations

a	Scalar a
$\mathbf{a}$	Vector $\mathbf{a}$
$\mathbf{A}$	Matrix $\mathbf{A}$
$(\cdot)^H$	Hermitian operator
$(\cdot)^{-1}$	Matrix inverse operator
$(\cdot)^{-H}$	Equivalent to $\left((\cdot)^H\right)^{-1}$
$\mathrm{E}\{\cdot\}$	Statistical expectation operator
$\mathrm{Tr}(\cdot)$	Trace operator
$\|\cdot\|_2$	Euclidean norm
$\|\cdot\|_F$	Frobenius norm
$\mathrm{Diag}(\cdot)$	Diagonal matrix with the arguments as the diagonal elements
$[\mathbf{A}]_{ij}$	(i,j)th element of $\mathbf{A}$
$[\mathbf{a}]_i$	ith element of $\mathbf{a}$
$\sigma_{\max}(\mathbf{A})$	Largest singular value of $\mathbf{A}$
$\lambda_i(\mathbf{A})$	ith eigenvalue of $\mathbf{A}$
$\lambda_{\min}(\mathbf{A})$	Smallest eigenvalue of $\mathbf{A}$
$\rho(\mathbf{A})$	Spectral radius of $\mathbf{A}$
$\mathbf{u} \geq \mathbf{v}$	$[\mathbf{u}]_i \geq [\mathbf{v}]_i \quad \forall i$
$\mathbf{A} \succeq \mathbf{0}$	$\mathbf{A}$ is positive semidefinite
$\mathbb{C}^{m\times n}$	Set of $m \times n$ complex matrices
$\mathbb{R}_{+}^{m\times n}$	Set of $m \times n$ matrices with real non-negative elements
$\mathbb{R}_{++}^{m\times n}$	Set of $m \times n$ matrices with real positive elements
$x \sim N(\mu, \sigma^2)$	Random variable x is drawn from a Gaussian distribution with mean μ and variance σ^2
$x \sim N_C(\mu, \sigma^2)$	Random variable x is drawn from a circularly symmetric complex Gaussian distribution with mean μ and variance σ^2
$U(a,b)$	Uniform probability distribution with interval $[a,b]$
$\mathcal{N}(\cdot)$	Null space operator
$\mathbf{P}_{\mathcal{N}(\mathbf{A})}$	Orthogonal projection onto the null space of $\mathbf{A}$
$(x)^+$	$\max(0,x)$
$[x]_a^b$	Euclidean projection of x onto the interval $[a,b]$
$[\mathbf{X}]_{\mathcal{Q}}$	$\arg\min_{\mathbf{Z}\in\mathcal{Q}} \|\|\mathbf{Z}-\mathbf{X}\|\|_F$, where $\mathcal{Q}$ is a convex set

Chapter 1

INTRODUCTION

Wireless communications technology has become an ubiquitous element of our society, ranging from remote controllers and paging systems to cellular phones and wireless local area networks. With the advent of better battery technology and the relentless progress of Moore's law, today's portable devices can support a great amount of processing power. This has led to an exponential increase in the usage of portable devices such as cellular phones, tablet computers and laptops for data-intensive broadband internet applications (Figure 1.1).

This huge increase in demand has led to heavy congestion in the radio-frequency (RF) spectrum allocated to these applications. Issues such as call dropping, low download speeds and sporadic availability of network access have become commonplace, particularly in areas with high density of users. This was exemplified at the launch of the popular iPhone 4 cell-phone last year, when Steve Jobs (CEO of Apple Inc.), who is famous for delivering impeccable product-launches, had to briefly suspend the launch midway as the demonstration phone could not access the network. He finally had to

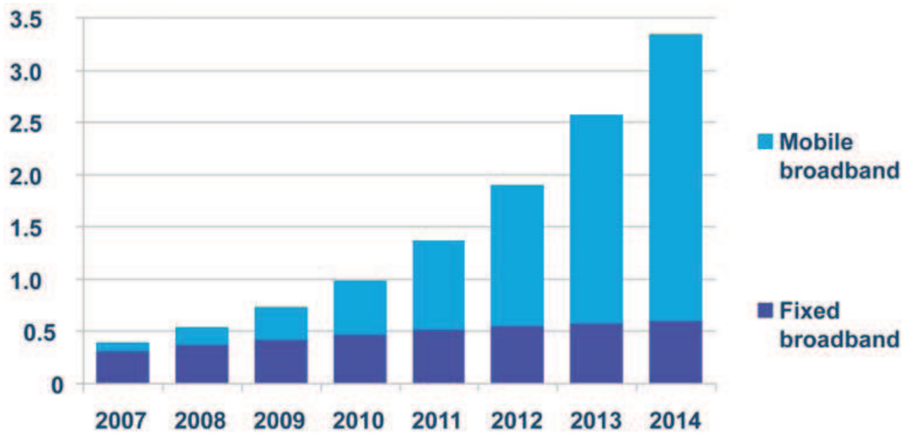


Figure 1.1: Projected number of subscriptions (in billions) for mobile broadband and wired broadband internet connections globally [1].

request the 500+ members of the audience to turn off their WiFi devices in order to continue the demonstration [2].

1.1 Radio frequency bands and spectrum management

Wireless communication devices for various applications and services operate in specific pre-determined ranges of the electromagnetic spectrum called *radio frequency (RF) bands*. The radio waves used for communication are transmitted and received through antennas which transform electrical energy to radio waves that propagate through the atmosphere from the source to the destination. Due to varying propagation characteristics of radio waves of different frequencies, specific applications are allocated specific RF bands. For instance, the RF spectrum allocation in the United States of America is presented in Figure 1.2. These allocations are determined by various national and international regulatory bodies (e.g.: the Federal Communications Commission (FCC) in the USA and the Office of Communications (Ofcom) in the UK).

When multiple wireless devices operate in the same environment, they often interfere with each other. *Spectrum management* is needed to control the usage of RF spectrum in order to mitigate interference among wireless devices and services. The current practice for spectrum allocation by regulatory bodies is known as the command-and-control model [4]. In this approach, the regulators make centralized decisions regarding spectrum allocation and usage, often through an auctioning process commonly referred to as a *spectrum auction*. After a successful bid, a user/company is awarded the allocation, which is often valid for extended periods of time and over large geographical regions. While the majority of the RF spectrum is managed under this scheme, a small region of the RF spectrum, known as the industrial, scientific and medical (ISM) band, is unlicensed and open to any device/application. Some of the technologies using these band are cordless telephony, bluetooth radio, wireless local area networking and radio-frequency identification (RFID).

The command-and-control model of spectrum management and allocation ensures interference-free operation for the licensed user as it is operating in the band exclusively. Since most of the spectrum is already allocated to various applications (Figure 1.2), emerging wireless applications such as wireless broadband communications face an apparent spectrum scarcity.

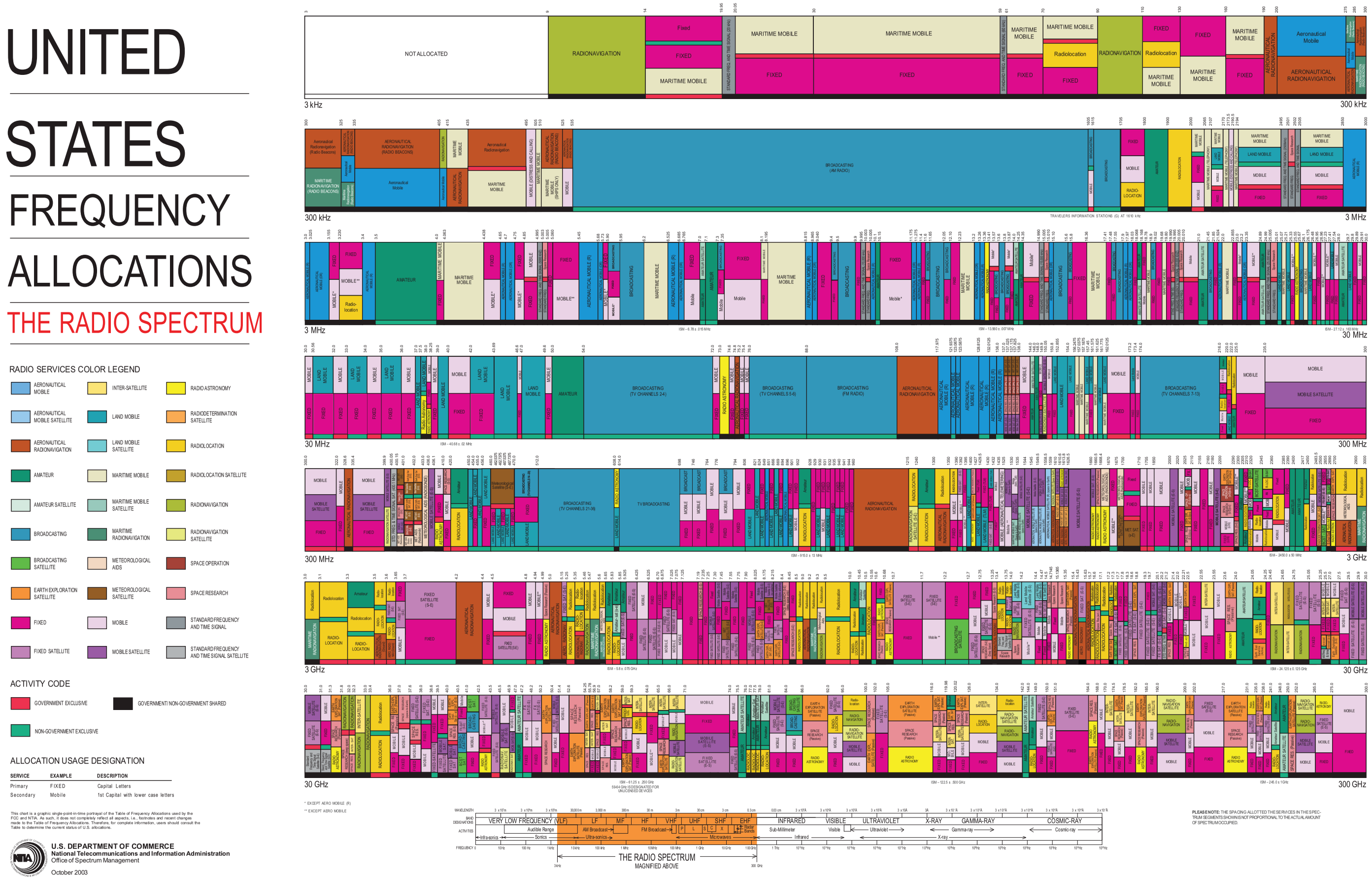


Figure 1.2: A visual representation of the congestion in radio spectrum allocation in the United States of America; the detail is unnecessary, but can be found at [3].

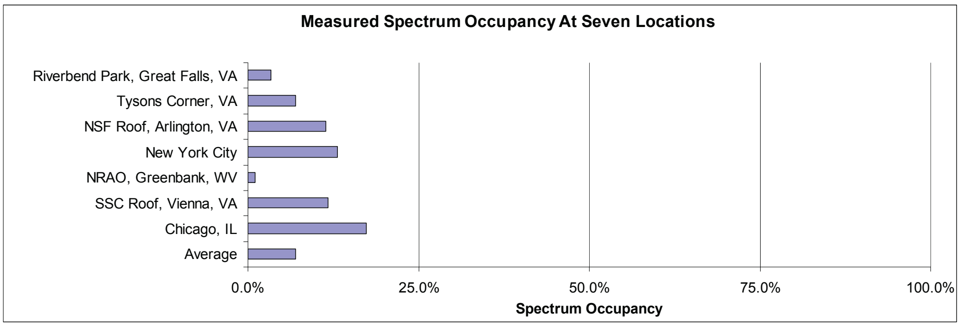
However, field studies on actual spectrum usage have shown that the current spectrum management policy results in highly inefficient spectrum utilization (Figure 1.3). This has led to the evolution of new paradigms and technologies for future spectrum management models and wireless communication devices which aim to improve the efficiency of their own spectrum utilization and to exploit the inefficiency of licensed spectrum users.

1.2 Emerging wireless paradigms and technologies

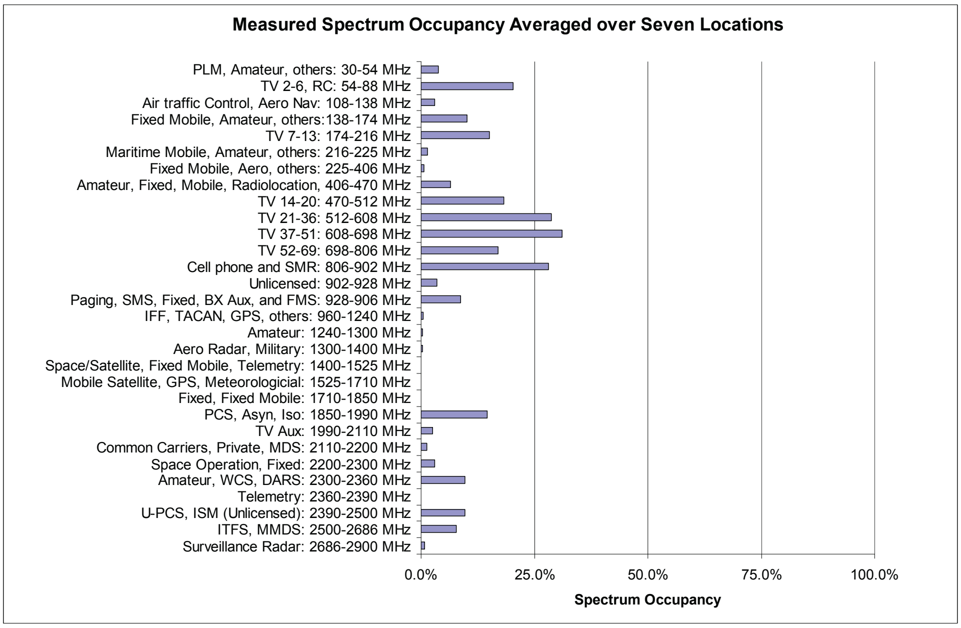
The growing demand for high-throughput wireless communication has led to the emergence of new paradigms and technologies as contenders for next-generation wireless communication networks. In this section, a few of these, namely cognitive radio, dynamic spectrum access, orthogonal frequency-division multiplexing (OFDM) and multi-antenna systems are briefly described.

1.2.1 Cognitive radio and dynamic spectrum access

Cognitive radio is an emerging paradigm of wireless communication in which an intelligent wireless system utilizes information about the radio environment to adapt its operating characteristics in order to ensure reliable communication and efficient spectrum utilization [6]. The main goal of this paradigm is to enable the cognitive radio to exploit the inefficiently utilized licensed spectrum for its own communication needs without significantly affecting the licensed user. The band of RF spectrum that is licensed to a user



(a) Overall spectrum occupancy measured at seven locations



(b) Measured spectrum occupancy in the 30 MHz-3,000 MHz range averaged over the seven locations

Figure 1.3: Average spectrum occupancy measured at seven locations in the USA in the 30 MHz-3,000 MHz range demonstrating inefficient spectrum utilization [5].

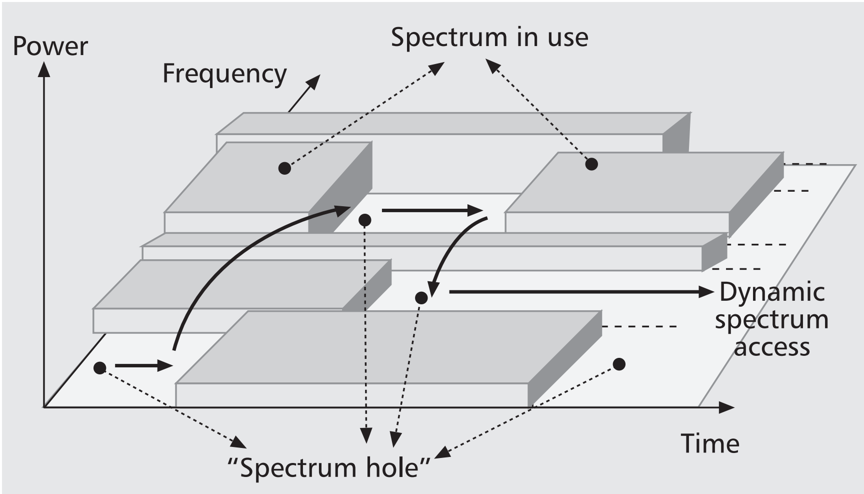


Figure 1.4: Concept of spectrum hole represented in power, time and frequency space wherein there is an opportunity for a cognitive radio to operate [7].

but is not utilized by the licensed user at a particular time and location is called a *spectrum hole* (Figure 1.4).

One of the key enabling technologies for cognitive radio is *dynamic spectrum access*. The overarching idea behind dynamic spectrum access is to temporarily borrow unused spectrum from licensed users (spectrum holes) without interfering with their operation [8]. Research on various aspects of cognitive radio and dynamic spectrum access has received a great deal of interest in recent times [9, 10, 11, 12, 13] and will help design wireless communication systems of the future.

1.2.2 OFDM technology

Orthogonal frequency-division multiplexing (OFDM) is a frequency-division multiplexing scheme developed to transmit multiple digital signals simultaneously over a large number of closely-spaced orthogonal sub-carriers [14]. OFDM transmission is used for wideband digital communication, both wired and wireless, and has been adopted for a variety of applications, ranging from digital television to wireless networking.

OFDM technology has many advantages. It has a high spectral efficiency and is resilient to interference and multipath effects. It can also easily adapt to severe channel conditions without complicated time-domain equalization and can be efficiently implemented using the fast Fourier transform. However, OFDM technology is quite sensitive to Doppler shift and frequency-synchronization issues, despite which it has become the de-facto physical (PHY) layer broadband transmission technology and is particularly advantageous for multiple access systems.

1.2.3 Multi-antenna technology

Multiple antennas at the receiver and/or transmitter of a wireless communication system can be used to improve link performance [15]. The term multiple-input multiple-output (MIMO) is used to describe systems which exploit such antenna diversity. MIMO technology can significantly improve the data throughput and coverage without additional bandwidth or transmission power. This is because signals transmitted from multiple antennas experience differing multipath fading and are received by multiple antennas. The different multipath signals can be combined coherently to achieve higher data rates and/or lower bit-error rates by using clever signal processing techniques. However, this can significantly increase the complexity of the communication system.

1.3 Motivation

The aforementioned technologies and paradigms are integral features of today's high performance networks and/or are strong contenders for wireless networks of tomorrow. OFDM and MIMO technologies are already extensively used in wireless networks through standards such as IEEE 802.11n and WiMAX. They are also strong candidate technologies for the next-generation of wireless communication networks [16, 17, 18], which will incorporate principles of cognitive radio and dynamic spectrum access [19, 20].

A prominent feature of these paradigms is the provision for greater freedom of action for the users in the network. In such a network, "intelligent" users actively and dynamically manage the resources and characteristics of their transceivers, such as transmit power and bandwidth, in order to optimize their communication performance in terms of various criteria such as information rate, utilized power and achieved quality-of-service (QoS). With the advent of cognitive radio and dynamic spectrum access, multiple heterogenous wireless technologies and standards of tomorrow are expected to function seamlessly in the same environment and RF spectrum [21]. In such a setting, interference among the users[1] becomes a critical issue and precludes the traditional approach of interference mitigation through predetermined non-overlapping frequency allocation (FDMA) due to significant demands on coordination among the users. This form of *centralized control* determining the communication parameters of users leads to heavy signalling overhead as information from all the users needs to be collected, processed and disseminated by the controller.

The issue of maximizing information rates of users in an interference channel by optimizing the power spectral density of the transmitted signal under certain power constraints is of interest in this thesis. The sum-rate maximization problem in a frequency-selective Gaussian interference channel has been proved to be NP-hard [22] and the optimal solution of this problem is of the form of frequency division multiple access (FDMA) [23]. Thus, a centralized solution to this problem not only requires information (in the form of channel state information and noise variances) from all users, but also is computationally unattractive.

The solution to these limitations lies in the framework of *distributed algorithms*, which enable users of the network to compute their optimal solutions autonomously with limited (locally available) information. The analysis of such a system of multiple interactive autonomous intelligent users operating in the same environment falls well within the purview of game theory, which

[1] Such a system with *multi-user interference* in a frequency-selective medium (as seen in OFDM transmission) can be modelled as an *interference channel*, described in Section 3.3 on page 61.

was devised to analyze and predict the outcome of situations where multiple entities (typically people, companies and nations) interact. In addition, solutions from such a game-theoretic analysis can often be implemented as distributed algorithms. These factors make game theory an attractive tool for the design and analysis of tomorrow's wireless communication networks. Indeed, it has been shown that the problem of maximizing information rates of users in an interference channel, which is of interest in this thesis, can be modelled as a noncooperative game [24].

However, a vast majority of game-theoretic solutions proposed for wireless networks in the current literature, including the rate-maximization game, assumes perfect knowledge of channel state information, which is not possible in practice, where such information is estimated with a certain degree of uncertainty. This uncertainty could be introduced though several mechanisms, such as estimation errors, feedback quantization and latency between channel estimation and signal transmission. If these solutions are to be implemented in practice, **the effect of such uncertainty on the performance of these game-theoretic solutions needs to be characterized and *robust* game-theoretic algorithms which perform satisfactorily in spite of such uncertainty need to be designed and analyzed.** This issue of uncertainty in the parameters of game-theoretic solutions is the central theme of this thesis.

In summary, **the design of distributed algorithms based on ideas from game theory for maximizing the information rates of users**

having limited transmit power in single-antenna and multi-antenna interference channels with uncertainty in channel state information is the focus of this thesis.

1.4 Thesis outline

The aforementioned approaches and issues are addressed over the following chapters of this thesis:

Chapter 2: Game Theory: Fundamentals, Nash Equilibrium and Robust Game Theory

This chapter presents a brief description of the game-theoretic concepts that are of interest in this thesis. The chapter begins with an introduction to game theory and the conditions needed for its application in a given scenario, described in Section 2.1 on page 35. This is followed by a discussion on strategic noncooperative games and the concept of Nash equilibrium in Section 2.2 on page 40 and the notion of equilibrium efficiency as a method to quantify the quality of the Nash equilibrium solution in Section 2.3 on page 42.

In Section 2.4 on page 45, a discussion on some of the limitations of the Nash equilibrium concept and the traditional game-theoretic approach to the issue of uncertainty in games is presented. Finally, the robust game model, which is a union of ideas from robust optimization theory and game theory, is introduced in Section 2.5 on page 47 as a suitable candidate for resolving the issue of uncertainty in channel state information affecting the performance of game-theoretic solutions.

Chapter:3: Iterative Waterfilling Algorithms

This chapter presents a specific game-theoretic solution and its associated conceptual and mathematical foundations within which the issue of channel uncertainty is investigated in this thesis. This includes results from contraction and fixed point theory, which addresses the formulation and characterization of distributed algorithms (Section 3.1 on page 52) and information theory, which introduce the waterfilling solution as the optimal solution to the problem of rate-maximization in a single-user context (Section 3.2 on page 57). This is followed by the description of the Gaussian interference channel in the multi-user scenario and a review of the current literature using game theory to address the problem of rate-maximization in this medium in Section 3.3 on page 61.

The predominant game-theoretic solution, namely the iterative waterfilling algorithm (IWFA), to the problem of rate-maximization in single-antenna frequency-selective Gaussian interference channels and multi-antenna (MIMO) Gaussian interference channels are presented respectively in Section 3.4 on page 66 and Section 3.5 on page 72 and extended to broadband MIMO Gaussian interference channels in Section 3.6 on page 80.

Finally, in Section 3.7 on page 85, the effect of channel state information errors on the performance of the MIMO iterative waterfilling algorithm is investigated, which demonstrates the need for robust solutions presented in the subsequent chapters.

Chapter 4: Robust IWFA for SISO Frequency-Selective Systems

In this chapter, the analytic framework for a robust formulation of the rate-maximization game for single-input single-output (SISO) frequency-selective Gaussian interference channels is presented. Section 4.1 on page 93 is a review of the state-of-the-art in methods that address and investigate the issue of channel state information uncertainty in rate-maximization games for Gaussian interference channels. The system under consideration is described in Section 4.2 on page 95. A distribution-free robust rate-maximization game based on the robust game model is formulated in Section 4.3 on page 98. The optimal solution of each user in the form of a robust waterfilling operation is derived and characterized in Section 4.4 on page 100.

In Section 4.5 on page 104, the equilibrium solution of this game, termed the *robust-optimization equilibrium*, is presented and shown to exist for possible channel values and initializations, and further, to be unique under certain sufficient conditions. A distributed algorithm to compute the equilibrium solution iteratively is presented and proved to asymptotically converge when a unique equilibrium is guaranteed in Section 4.6 on page 105. Finally, numerical simulations to confirm the behaviour of the algorithm are presented in Section 4.7 on page 108, where an interesting effect of increase in sum-rate with greater channel uncertainty is observed, which merits further analysis.

Chapter 5: Sum-Rate Analysis in the Two-User Scenario

This chapter is an analytical investigation of the improvement in sum-rate with greater channel uncertainty in the robust SISO rate-maximization game proposed in Chapter 4 for the two-user case. To begin with, the effect of increasing channel uncertainty on the sum-rate and the price of anarchy of a simple two-frequency system are analyzed in Section 5.1 on page 124. Based on these results, conditions for improvement in the sum-rate and the price of anarchy of a system with asymptotically large number of frequencies with an increase in uncertainty are derived in Section 5.2 on page 127. Finally, these results are supported using simulations in Section 5.3 on page 129.

Chapter 6: Robust IWFA for MIMO Systems

In this chapter, a robust rate-maximization game in MIMO Gaussian interference channels in the presence of bounded channel uncertainty is developed. The system model for which the robust game is developed is described in Section 6.1 on page 151. A robust MIMO rate-maximization game for this system is formulated and shown to be a modified MIMO rate-maximization game (Section 3.5) in Section 6.2 on page 153. The robust-optimization equilibrium for this game and an iterative waterfilling algorithm to compute it are presented, along with sufficient conditions for the uniqueness of the equilibrium and asymptotic convergence of the algorithm, in Section 6.3 on page 157. The behaviour of the algorithm under different settings is confirmed though numerical simulations in Section 6.4 on page 161.

Chapter 7: Summary, Conclusions and Future Work

The novel results of this thesis and their conclusions are summarized in Section 7.1 on page 167. The work presented in this thesis could be extended in various directions, some of which are described in Section 7.2 on page 171.

Chapter 2

GAME THEORY — FUNDAMENTALS, NASH EQUILIBRIUM AND ROBUST GAME THEORY

The past decade has seen the increasing application of concepts from game theory in wireless communication systems to solve a variety of problems [25, 26, 27, 28, 29, 30, 31]. Game theory has been utilized to solve various resource allocations problems in different scenarios and the problems considered involve bandwidth allocation, power control, medium access control, flow control, routing and pricing issues in wireless networks [32]. Game theory has also been extensively applied in cognitive radio and dynamic spectrum access, particularly to solve issues in spectrum management and spectrum sharing [33, 34, 35, 36, 37].

In this chapter, a brief overview of game-theoretic concepts relevant to this thesis is presented. The chapter begins with an introduction to game theory and a brief discussion on the classification of games. This is followed by a description of the strategic noncooperative game model, the Nash equilibrium and the notion of equilibrium efficiency. Finally, the robust game model is introduced as a way of moving beyond the concept of the Nash equilibrium in static noncooperative games having uncertainty in payoff functions, with the marriage of ideas from robust optimization theory and game theory.

2.1 Introduction to game theory

Game theory is a collection of mathematical tools designed for the analysis of situations where decision-makers meet and interact. In such situations, the success of individual decision-makers depends on the actions of others. Though game theory was developed to analyze and understand economic behaviour [38, 39], the underlying concepts of game theory are far-reaching and have seen applications in diverse fields such as biology, political science, international relations, computer science, engineering, social psychology, philosophy and management.

In its broadest sense, a *game* is a description of strategic interaction among decision makers, termed as *players*. It specifies the constraints on the players when deciding on a possible action, but does not state what action they decide to take. A *solution concept* of a game refers to a formal rule that predicts the actions of the players when the game is played. These predictions are called *solutions* and describe the action each player will select, which is called a *strategy*. The most common form of the solution concept is the *equilibrium concept*. At an equilibrium, the various forces influencing the game are balanced and the players will not change their strategies in the absence of external influences.

Game theory is built upon the *theory of rational choice*, which assumes that the action decided by a player is at least as good as every other available action. The players are also assumed to reason strategically by taking into account the possible actions of other players while deciding their own actions.

In order to apply game theory to analyze a given scenario, a few basic requirements need to be satisfied. First of all, there must be well-defined decision making processes in the scenario. Also, the decisions from these processes should have a predictable impact on performance. This is typically achieved by clearly identifying the players of the game and specifying their preferences explicitly through utility/payoff functions which are mappings from the set of possible actions to the profits delivered. Furthermore, in order to ensure that the scenario being modelled does not result in a trivial game, there must be multiple *interactive* decision makers and each decision maker must have multiple possible actions. The term "interactive" in this context indicates that the actions of any decision maker should have an impact on the actions of the others. In other words, the scenario being modelled as a game should not be a simple single-objective optimization problem. In most cases, having a multi-user scenario with a separate objective function for each user will be adequate for this to be ensured.[1]

Traditionally, game theory was developed in order to analyze and predict the outcomes of situations, particularly in economics, wherein multiple decision makers interacted, given their decision making processes. In a communications network context, this implies that applying game theory in this fashion will help to predict the performance of networks and to analyze the effect of various network parameters on the network performance when the users are operating under a certain protocol or utilizing a specified algorithm.

In engineering (and particularly in communications), there is a great range of decision making processes and possible actions that could exist in a certain scenario (system), unlike in the case of economics or political science where people, companies and nations are being modelled. Further, in the case of economics or political science, the game-theorist has little control or influence on the actions of the set of people, companies or nations being modelled and game theory is primarily an analytical tool in such cases. However, there could be certain specific decision making processes and ranges of actions whose outcome predicted by game theory is more desirable than others, which indicates that game theory could also be used as a design tool. In a

[1]This does not necessarily mean that game theory is only applicable in multi-user problems. In some cases, inventing fictitious users playing an imaginary game may be useful in achieving a desired solution. For instance, a single-user robust (worst-case) power allocation problem in MIMO channels with no channel information or statistics has been analyzed by modelling the problem as a user vs. malicious-nature game [40] . In this case, the interaction between the user and nature occurs through the varying levels of receiver noise of the user.

communications network setting, this translates to designing (engineering!) the protocols or algorithms utilized by the users in the network in order to achieve specific predetermined performance targets at the user and/or network level.[2] It is this "engineering" perspective of applying game theory as a design tool that is of great interest in the field of wireless communications and networking.

2.1.1 Types of games

A *game* is a mathematical model of interacting decision makers and has three basic components [42]:

- a set of players
- a set of actions (for each player)
- a set of preferences (for each player)

The set of actions of each player is called the *set of admissible strategies* or the *strategy space*. The preferences of each player are typically specified using *payoff/utility functions* which explicitly describe the relation between actions and profits.

There is a huge diversity in game-theoretic approaches and classifying this seemingly bewildering variety of games under a universal classification

[2]This is quite similar to the concept of reverse engineering [41], where the goal is to discover/reinvent the process that resulted in a solution/product, given the final solution/product, through the analysis of its structure, function and operation.

scheme is not easy. This has led to many classifications of game-theoretic methods along different criteria, some of which are:

- **Cooperative and noncooperative games:** If the players in a game are aiming to mutually benefit by reliably cooperating with each other, this leads to cooperative games. On the other hand, if the users are aiming to improve only their own selfish objectives, the game is said to be noncooperative.

- **Static and dynamic games:** These are also known as *strategic games* and *extensive games*, respectively. In static games, all players make their decisions simultaneously (or if at different points in time, without knowing other players' strategies), whereas in dynamic games, there is a temporal component where the players may take turns to make decisions or play the game repeatedly and try to take advantage of knowing the history of the game.

- **Zero-sum and non-zero-sum games:** In zero-sum games (and more generally, constant-sum games), the total available resources in the game is constant and unaffected by the players' actions, with the gain by any player being offset by a corresponding loss of others. However, in non-zero-sum games, the total available resources in the game is not constant and depends on the strategies of the players. In such cases, a careful design of the game may lead to improvement of many or all the payoffs of the players in the game.

- **Symmetric and asymmetric games:** In symmetric games, all the players have identical payoff functions and strategy-spaces and the payoff for a particular strategy is independent of the identity of the player. However, in asymmetric games this is not the case, where the payoff functions and/or strategy-spaces are non-identical and the payoffs of the players are non-identical.

The games considered in this thesis are static noncooperative asymmetric non-zero-sum games. In the following section, a brief description of the strategic noncooperative game and its solution concept, the Nash equilibrium, is presented.

2.2 Static noncooperative game — Nash equilibrium

This section formalizes the strategic noncooperative game model and the definition of the Nash equilibrium solution concept.[3]

Consider the following static noncooperative game, $\mathscr{G}$:

- Set of players: $\Omega = \{1, \dots, Q\}$.
- Set of admissible strategies (Strategy-space): $\mathcal{A}_q \neq \{\ \}\ \ \forall q \in \Omega$.
- Payoff (Utility) functions: $U_q(a_q, \mathbf{a_{-q}}) : \mathcal{A}_1 \times \cdots \times \mathcal{A}_Q \mapsto \mathbb{R}\ \ \forall q \in \Omega$

[3]The discussion presented here has been limited to *pure strategies*, i.e., actions that are deterministic. The notions presented here have been extended to *mixed strategies*, where the pure strategies are associated with a probability of application. However, as all the games discussed in this thesis have pure strategies, game-theoretic concepts involving mixed strategies are beyond the scope of this thesis. Refer to [42, 43] among others for further information on mixed strategies.

where $a_q \in \mathcal{A}_q$ and $\mathbf{a}_{-q} = \{a_i\}_{i \neq q}$.

where each player is assumed to know the strategy-space and payoff functions of every other player. This knowledge is not always necessary in all noncooperative games. For instance, in the robust MIMO rate-maximization game analyzed in Section 6.2 on page 153, the players do not explicit share any information.

The solution concept of this game is the famous *Nash equilibrium*, based on the concepts introduced by Nobel Prize winner John Nash in [44,45]. The notion of a Nash equilibrium is presented in the following result:

Proposition 2.1. *Given the game* $\mathscr{G}$*, the joint strategy* $\mathbf{a}^* \triangleq [a_1^*, \dots, a_Q^*]$ *is a Nash equilibrium if*

$$U_q(a_q^*, \mathbf{a}_{-q}^*) \geq U_q(a_q, \mathbf{a}_{-q}^*) \quad \forall a_q \in \mathcal{A}_q, \quad \forall q \in \Omega. \tag{2.2.1}$$

In other words, no single player can profit by unilaterally deviating from a Nash equilibrium. An alternate interpretation of the Nash equilibrium is through the concept of *best response strategies*. A best response strategy of a player is the action (or set of actions) which results in the most favourable outcome for a player, given other players' strategies. At the Nash equilibrium, each player's strategy is a best response to all other strategies in the equilibrium. Thus, the Nash equilibrium of a game can be said to be a joint-best-response strategy of the players.

The question of whether a given game has any equilibrium or not is one that has been extensively investigated over the years. Most of these are built upon various fixed point theorems,[4,5] and present various conditions under which a game will be guaranteed to have an equilibrium (Refer to [50] for a detailed treatment on the topic). A further question to be addressed is whether there is a unique Nash equilibrium in a game, and if so, ways to compute it. This is because the equilibrium solution concept only indicates that if the players are initialized with a Nash equilibrium solution, then they will continue to remain there. It does not specify the dynamics of the game, and how long the players may take to converge to an equilibrium, if they converge at all.

2.3 Equilibrium efficiency

The "joint-best-response" interpretation of the Nash equilibrium is of significant interest when dealing with distributed optimization. It implies that each player is at a locally optimal solution which can be computed by a distributed algorithm whose convergence properties could be characterized and analyzed. This leads to the idea of *competitive optimality*, where each player in a competitive environment settles down to a (locally) optimal *stable* solution. The term "optimality" here is slightly fallacious and this idea

[4]It is interesting that the equivalence between the Nash equilibrium and the concept of a fixed point is only mentioned in passing, if at all, in most general textbooks on game theory, when John Nash's work is actually built on this interpretation, using Kakutani's fixed point theorem [46] in his seminal paper [44] to prove the existence of an equilibrium.

[5]A few of the important fixed point theorems in game theory are the ones by Brouwer [47], Lefschetz [48], Hopf [49] and Kakutani [46].

should be applied with caution. This is because the emphasis in such an approach is on achieving a stable (stationary) situation in the game with all the players settling to a single action each, rather than an emphasis on optimizing the total (or individual) utility of the players in the game. In fact, there is no guarantee that such a decentralized noncooperative solution will yield a utility as good as one from a centralized optimization approach. This leads to the concept of *equilibrium efficiency* which tries to quantify this issue and measure the trade-off between achieving decentralized control and having globally optimal solutions.

The popular measures of equilibrium efficiency are as follows:

1. Social welfare
2. Pareto optimality
3. Price of anarchy/stability

2.3.1 Social welfare

Given the joint strategy $\mathbf{a} \triangleq [a_1, \ldots, a_Q]$, the social welfare of game $\mathscr{G}$ is defined as the sum of all the utilities of the players

$$w(\mathbf{a}) = \sum_{q=1}^{Q} U_q(a_q, \mathbf{a}_{-q}) \tag{2.3.1}$$

When the payoff functions of the players are the information rates of the users (as is the case in this thesis), the social welfare is equivalent to the sum-rate of the system.

2.3.2 Pareto optimality

A solution is Pareto optimal if there is no possible way to improve the payoff of a (non-empty) subset of players without leaving any other player worse-off.

Proposition 2.2. *Given the game $\mathscr{G}$, a joint strategy $\mathbf{a}^* \triangleq [a_1^*, \ldots, a_Q^*]$ is said to Pareto optimal if there does NOT exist any $\mathbf{a} \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_Q$ such that*

$$\mathbf{U}(\mathbf{a}) > \mathbf{U}(\mathbf{a}^*) \tag{2.3.2}$$

where $\mathbf{U}(\mathbf{a}) \triangleq [U_1(a_1, \mathbf{a}_{-1}), \ldots, U_Q(a_Q, \mathbf{a}_{-Q})]$.

The set of solutions which are Pareto optimal forms the boundary of the joint-utility region of the game and is known as the Pareto frontier [51].

2.3.3 Price of anarchy and price of stability

The concept of price of anarchy, introduced in [52], aims to measure the "price of uncoordinated individual utility-maximizing decisions". Together with the price of stability, it helps quantify the trade-off between having distributed algorithms and optimal social welfare.

The price of anarchy is defined as the ratio between the objective function value at the socially optimal solution and the *worst* objective function value at any equilibrium of the game [53]. The price of stability is defined as the ratio between the objective function value at the socially optimal solution and the *best* objective function value at any equilibrium of the game [53].

Thus, in game $\mathscr{G}$,

$$
\begin{aligned}
\text{PoA} &= \frac{\max_{\mathbf{a}\in\mathcal{A}} \; w(\mathbf{a})}{\min_{\mathbf{a}\in\mathcal{A}_{NE}} \; w(\mathbf{a})} \\
\text{PoS} &= \frac{\max_{\mathbf{a}\in\mathcal{A}} \; w(\mathbf{a})}{\max_{\mathbf{a}\in\mathcal{A}_{NE}} \; w(\mathbf{a})}
\end{aligned}
\tag{2.3.3}
$$

where $\mathcal{A} \triangleq \mathcal{A}_1 \times \cdots \times \mathcal{A}_Q$ is the joint strategy space, $\mathcal{A}_{NE}$ is the set of Nash equilibrium solutions and $w(\mathbf{a})$ is the social welfare of the game as defined in (2.3.1). Note that PoA $\geq$ PoS ≥ 1 and that a lower price of anarchy indicates a more efficient game. Thus, price of anarchy and price of stability help quantify the worst-case and best-case equilibrium efficiency respectively. In games with unique Nash equilibrium solutions, such as the ones considered in this thesis, the price of anarchy and stability are identical.

2.4 Moving beyond Nash equilibrium — robust game theory

The Nash equilibrium concept discussed in the previous section is one of the cornerstones of noncooperative game theory and has been utilized to solve a great many problems in a diverse range of applications. However, the concept of Nash equilibrium is not without limitations. As seen in the previous section, the Nash equilibrium concept generally provides a distributed solution at the cost of equilibrium efficiency. However, it suffers from scalability issues. In games with large number of players and large action spaces, the computation of the Nash equilibrium solution and the verification of the conditions for existence and/or uniqueness often become inefficient or even intractable. Furthermore, these costly computations to evaluate the equilibrium solution may not even be Pareto optimal.

The scalability issue is even more complicated in large games as they typically yield large number of equilibria and could result in a high price of anarchy. In such a scenario, it is quite difficult to predict which equilibrium is achieved or to ensure that a specific subset of equilibria is achieved as different initializations could lead to different equilibria. Techniques to identify and select an appropriate equilibrium in such cases lead to the theory of equilibrium selection [54]. The issue of multiple equilibria is not of importance in this thesis as the focus of the methodologies here are in ensuring a unique equilibrium and ways to compute it easily in a distributed fashion, which is presented in Section 3.1 on page 52. Scalability issues in large games are beyond the scope of this thesis and is a possible avenue of future research.

Another limitation of the concept of Nash equilibrium is that it assumes complete knowledge of all the players' actions by each player and that these actions and the payoffs of each player are known accurately. However, this might not be possible in many cases and the players are often uncertain about some aspects of the game. Uncertainty in the payoffs of the players is of particular interest in this thesis, as perfect knowledge of channel state information is not available in wireless networking games. Thus, alternate equilibrium concepts which model the presence of uncertainty need to considered.

The traditional game-theoretic solution to the uncertainty issue is Harsanyi's Bayesian game model [55,56,57] which is essentially analogous to the stochastic programming approach [51] to uncertainty in data-sets in optimization theory. In the Bayesian game model, each player aims to maximize the expected payoff, given the full prior probability distribution of all the parameters with uncertainty. This model assumes that all the players have the same prior probability distribution and is known to all the players. Although this model has been extended to relax the assumptions of common prior and common knowledge, the drawback of this approach is that it is difficult, in practice, to estimate the prior probability distributions accurately and often leads to complicated probability distributions which results in the analysis of the game being intractable. Thus, it is desirable to move to novel "distribution-free" game models, which are defined to be independent of the prior probability distributions of the parameters with uncertainty.

2.5 Robust game model

The robust game model, proposed independently in [58,59], incorporates the concept of robust optimization [60], which is independent of the probability distribution of the parameters with uncertainty, into the framework of static noncooperative game theory. This approach models incomplete-information games as distribution-free *robust games* where the players use a worst-case robust optimization approach to counter bounded payoff uncertainty. The solution of this robust game model is a distribution-free equilibrium concept called the *robust-optimization equilibrium*.[6] In [58], the robust game model is proposed for finite N-person games with linear payoff functions and uncertainty in the parameters of the payoff function. The robust game model proposed in [59] is for two-person games (bimatrix games) with linear payoff functions and uncertainty in the actions of the opposing player and each player's payoff function parameters. This model has been extended in [61] to N-person games with nonlinear payoff functions, where it is reformulated as second-order cone complementary problems in order to solve certain classes of games. The work presented in this thesis is based on the robust game model in [58].

In worst-case robust optimization [51], the parameters are assumed to belong to "uncertainty sets" (set of all possible parameter values), and the objective function is optimized for the worst-case parameter value (i.e., the parameter value which results in the worst objective function value). **In the robust game model, each player formulates a best response as the solution to a worst-case optimization problem.** It is to be noted that the players apply a worst-case perspective *only* to the uncertain parameters that define their own payoff functions, given the actions of the other players, and that the actions of other players are beyond the scope of consideration of each player. In other words, the optimization performed by each player is for

[6] The equilibrium concept is called "robust Nash equilibrium" in [59] and "robust-optimization equilibrium" in [58]. In this thesis, the term robust-optimization equilibrium is used exclusively in order to highlight the fact that this concept has its roots in robust optimization theory.

worst-case payoff function parameters (presumably determined by “nature”) and *not* for worst-case actions of the other players.

If it is commonly known to all players that each one of them is adopting the above robust-optimization approach to payoff function uncertainty, then it is possible for the players to mutually predict each other’s behaviour, similar to the complete-information game whose solution is the Nash equilibrium.[7] The players reach an equilibrium when their mutual predictions coincide and this leads to the notion of the robust-optimization equilibrium.

This robust game approach is utilized in Chapters 4 and 6 to formulate a robust rate-maximization game in the presence of bounded channel uncertainty in SISO frequency-selective and MIMO Gaussian interference channels (For a discussion on Gaussian interference channels, refer to Section 3.3 on page 61).

2.6 Summary

This chapter presented an overview of the various concepts and methodologies from game theory that are of interest in the thesis. A brief introduction to game theory and the underlying assumptions needed to apply game theory were discussed. This was followed by a review of some of the commonly

[7]Since the players of the complete-information noncooperative game are rational and know the payoffs and action-space of each other, each player can compute the best responses of every other player, which will help predict the actions of the other users. Based on these predictions, each player formulates a best response strategy. The Nash equilibrium is the set of all mutually-coinciding predictions of the players. This idea may not be not straightforward to apply in practice, where all the information may not be available and multiple equilibria could exist.

observed types of games. Following this, the strategic noncooperative game model and the Nash equilibrium solution concept were formally defined and explained. The concept of equilibrium efficiency and measures to quantify it were then presented. The subsequent section elaborated some of the limitations of the Nash equilibrium and reviewed the traditional game-theoretic approach to address the issue of uncertainty. Finally, the concept of robust game theory as an amalgamation of approaches from robust optimization theory and game theory was explored.

Chapter 3

ITERATIVE WATERFILLING ALGORITHMS

This chapter presents the conceptual foundations and specific game-theoretic problem formulations on which the contributions of this thesis are based. The chapter starts with a brief overview of relevant results in contraction mappings and fixed point theory. This is followed by a brief summary of classical single-user waterfilling solutions. Next, the issue of rate-maximization in multi-user Gaussian interference channels is considered and the predominant game-theoretic approach to competitive rate-maximization in single antenna and multi-antenna Gaussian interference channels is outlined. The effect of channel estimation errors on the performance of this method is then investigated, setting the stage for the robust solutions proposed in this thesis in subsequent chapters. Finally, a short appendix on Karush-Kuhn–Tucker conditions is included for completeness.

3.1 Contraction and fixed point theory

In this section, some of the central concepts involved in solving nonlinear problems using distributed iterative algorithms are summarized (Refer to [62] for a rigorous treatment of the topic). The key motivation behind using these well-studied analytical tools lies in the interpretation of the Nash equilibrium as a fixed point and the representation of the waterfilling function as an Euclidean projection, enabling its interpretation as a contraction mapping.

3.1.1 Existence and uniqueness of a fixed point

Let $\mathbf{F} : \mathcal{X} \mapsto \mathcal{X}$ be any mapping from a subset $\mathcal{X} \subseteq \mathbb{R}^n$ to itself which is associated to a dynamic system described by

$$\mathbf{x}(n+1) = \mathbf{F}(\mathbf{x}(n)), \qquad n \in \mathbb{N}_+ = \{0, 1, 2, \dots\}, \tag{3.1.1}$$

where $\mathbf{x}(n) \in \mathbb{R}^n$ is the state variable vector at discrete-time n, with $\mathbf{x}(0) \in \mathbb{R}^n$. If this mapping has the property

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| \leq \alpha \|\mathbf{x} - \mathbf{y}\|, \qquad \forall\, \mathbf{x}, \mathbf{y} \in \mathcal{X}, \tag{3.1.2}$$

with $\|\cdot\|$ being some norm and α being a constant in the interval $[0, 1)$, then such a mapping is called a *contraction mapping*. The scalar α is called the *modulus* of $\mathbf{F}$. A mapping $\mathbf{F} : \mathcal{X} \mapsto \mathcal{Y}$ where $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$ that satisfies (3.1.2), is also called a contraction mapping, even if $\mathcal{X} \neq \mathcal{Y}$.

Any vector $\mathbf{x}^* \in \mathcal{X}$ satisfying $\mathbf{x}^* = \mathbf{F}(\mathbf{x}^*)$ is called a *fixed point* of the mapping and the relation (3.1.1) can be seen as an iterative algorithm to compute such a fixed point. This is possible because $\mathbf{x}^*$ is a fixed point if $\mathbf{F}$ is continuous at $\mathbf{x}^*$ and the sequence $\{\mathbf{x}(n)\}$ converges to $\mathbf{x}^*$.

The following result provides the conditions for the existence and uniqueness of such a fixed point [63]:

THEOREM 3.1. *Given the dynamic system in* (3.1.1) *with* $\mathbf{F} : \mathcal{X} \mapsto \mathcal{X}$ *and* $\mathcal{X} \subseteq \mathbb{R}^n$,

Existence: *If* $\mathcal{X}$ *is nonempty, convex and compact*[1], *and* $\mathbf{F}$ *is a continuous mapping, then there exists some* $\mathbf{x}^*$ *such that* $\mathbf{x}^* = \mathbf{F}(\mathbf{x}^*)$;

Uniqueness: *If* $\mathcal{X}$ *is closed and* $\mathbf{F}$ *is a contraction mapping in some vector norm* $\|\cdot\|$, *with modulus* $\alpha \in [0, 1)$, *then the fixed point of* $\mathbf{F}$ *is unique.*

It is noteworthy that the conditions in Theorem 3.1 are only sufficient conditions for the existence and uniqueness of a fixed point of a dynamic system. Further, this result, more specifically the contraction mapping, is norm-dependent. It is possible for mappings to be contractive under some norm and yet fail to be a contraction under a different norm. Thus, the choice of a suitable norm is critical in the application of this theorem. On the other hand, this flexibility in the choice of norm could lead to a more varied characterization of sufficient conditions for the existence and uniqueness of the fixed point of the mapping under different norms.

[1] A subset of the Euclidean space $\mathbb{R}^n$ is called compact if it is closed and bounded.

3.1.2 Convergence of distributed algorithms to a fixed point

The iterative algorithm described by $\mathbf{x} := \mathbf{F}(\mathbf{x})$ of dimension n can be transformed into a distributed algorithm whose components can be computed locally (at different times, if necessary) by suitably partitioning the system.[2] Let the partition of $\mathbf{x}$ be $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_Q)$ with $\mathbf{x}_q \in \mathbb{R}^{n_q}$ and $n_1 + \cdots + n_Q = n$ such that $\mathbf{F} = \{\mathbf{F}_q\}_{q=1}^{Q}$, with each $\mathbf{F}_q : \mathcal{X}_q \mapsto \mathcal{X}_q$ such that $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_Q$, with each $\mathcal{X}_q \subseteq \mathbb{R}^{n_q}$. According to this partition, the block-maximum norm on $\mathbb{R}^n$ is defined as

$$\|\mathbf{F}\|_{\text{block}} = \max_q \|\mathbf{F}_q\|_q \tag{3.1.3}$$

where $\|\cdot\|_q$ is any vector norm on $\mathbb{R}^{n_q}$ for each q. The mapping $\mathbf{F}$ is called a block-contraction with modulus $\alpha \in [0, 1)$ if it is a contraction in the block-maximum norm with modulus α.

Thus, the distributed implementation of the iterative algorithm (3.1.1) can be written as

$$\mathbf{x}_q = \mathbf{F}_q(\mathbf{x}), \qquad \forall\ q = 1, \ldots, Q. \tag{3.1.4}$$

The fixed point of $\mathbf{F}$, i.e., $\mathbf{x}^* = \mathbf{F}(\mathbf{x}^*)$ is equivalent to computing the fixed point of each component locally,

$$\mathbf{x}_q^* = \mathbf{F}_q(\mathbf{x}^*), \qquad \forall\ q = 1, \ldots, Q. \tag{3.1.5}$$

[2]Since the joint admissible strategy set of the games in this thesis is a Cartesian product of the set of admissible strategies of each player, the results presented here are limited to mappings whose domain can be written as a Cartesian product of lower dimensional sets.

Let $\mathbf{x}_q(n)$ be the value of the qth component at time n. Let the discrete set $\mathbb{T} \subseteq \mathbb{N}_+ = \{1, 2, \dots\}$ be the set of times at which one or more of the components are updated and $\mathbb{T}_q$ be the set of time instants n when the component $\mathbf{x}_q(n)$ is updated. Further, the most recent values of the other components may not be available during the computation of $\mathbf{x}_i$; thus, when $n \in \mathbb{T}_q$,

$$\begin{aligned}\mathbf{x}_q(n+1) = \mathbf{F}_q\Big(&\mathbf{x}_1(\tau_1^q(n)), \dots, \mathbf{x}_{q-1}(\tau_{q-1}^q(n)),\\ &\mathbf{x}_{q+1}(\tau_{q+1}^q(n)), \dots, \mathbf{x}_Q(\tau_Q^q(n))\Big),\end{aligned} \tag{3.1.6}$$

where $\tau_r^q(n)$ is the time of the most recent value of component r available to user q at time n.

Different update order and scheduling of the components of a distributed algorithm lead to different classes of algorithms. The most common of these are:

Jacobi scheme: All components $(\mathbf{x}_1, \dots, \mathbf{x}_Q)$ are updated *simultaneously*, via the mapping $\mathbf{F}$.

Gauss-Seidel scheme: All components $(\mathbf{x}_1, \dots, \mathbf{x}_Q)$ are updated *sequentially*, one after the other, via the mapping $\mathbf{F}$.

Totally asynchronous scheme: The components $(\mathbf{x}_1, \dots, \mathbf{x}_Q)$ are updated *fully asynchronously*, i.e., in no particular order or even with the same frequency, and the computation of some components may involve the use of outdated values of other components.[3]

The update order is of particular importance in the design of distributed algorithms. The convergence properties of the same mapping may differ significantly across different update schemes, possibly converging at different rates, to different fixed-points or even not converging at all. The Gauss-Seidel and Jacobi schemes are special cases of the totally asynchronous scheme. The algorithm model described and characterized in this section is totally asynchronous, which is the most general case of the classes described above, and thus most useful to help characterize a larger class of algorithms.

The following weak assumptions are made for each component q for the system to be totally asynchronous:

1. The system is causal.

$$0 \leq \tau_r^q(n) \leq n \tag{3.1.7}$$

2. Out-dated information is eventually purged.

$$\lim_{k\to\infty} \tau_r^q(n_k) = +\infty \tag{3.1.8}$$

3. No component fails to update its value eventually as time n progresses.

$$|\mathbb{T}| = \infty \tag{3.1.9}$$

[3]Variations of this scheme, such as having constraints on maximum tolerable delay, leads to a class of *partially* asynchronous algorithms.

These assumptions are generally satisfied in any practical implementation of distributed algorithms. The sufficient condition for convergence of a totally asynchronous distributed algorithm as described in this section is given in the following theorem [63]:

THEOREM 3.2. *Given the dynamic system in* (3.1.1) *with the mapping* $\mathbf{F} = \{\mathbf{F}_q\}_{q=1}^{Q} : \mathcal{X} \mapsto \mathcal{X}$ *with* $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_Q$*, if the mapping* $\mathbf{F}$ *is a block-contraction with modulus* $\alpha \in [0, 1)$*, then the totally asynchronous algorithm based on the mapping* $\mathbf{F}$ *asymptotically converges to the unique fixed point of* $\mathbf{F}$ *for any set of initial conditions in* $\mathcal{X}$ *and updating schedule.*

3.2 Waterfilling: classical results — single-user systems

In this section, the core concepts in information theory which led to the development of waterfilling as a solution to rate-maximization problems are presented.

Many common communication channels are modelled as a Gaussian channel, which is a time-discrete channel that models the noise at the receiver as an additive Gaussian random variable,

$$\mathbf{y} = \mathbf{x} + \mathbf{n}, \qquad \mathbf{n} \sim N_C(0, \sigma^2), \tag{3.2.1}$$

where $\mathbf{x}$ is the data transmitted in the current time-slot, $\mathbf{y}$ is the signal received and $\mathbf{n}$ is the complex noise drawn from a circularly symmetric complex Gaussian distribution of zero mean and variance σ^2.

If the average power constraint for transmission is P, then the capacity of the Gaussian channel with noise variance σ^2 is given by [64]

$$C = \frac{1}{2}\log\left(1+\frac{P}{\sigma^2}\right) \text{ nats/transmission.} \tag{3.2.2}$$

3.2.1 Parallel Gaussian channels

Now consider N independent Gaussian channels which transmit data in parallel with a common power constraint, P. This channel models a wideband non-white additive Gaussian noise channel, where each parallel component represents a different frequency. Let $\mathbf{p} \triangleq [p(1),\ldots,p(N)]$ where $p(k)$ is the power allocated to the kth channel and $\sigma^2(k)$ be the variance of the additive Gaussian noise in the kth channel. The goal is to allocate the power across the different channels in order to maximize the overall information rate. The solution to this *rate-maximization* problem, known as *waterfilling*, is given by the following result [64]:

Proposition 3.1. *The solution to the optimization problem*

$$\begin{aligned} \max_{\mathbf{p}} \quad & \sum_{k=1}^{N} \log\left(1+\frac{p(k)}{\sigma^2(k)}\right) \\ \text{s.t.} \quad & p(k) \geq 0, \\ & \sum_{k=1}^{N} p(k) = P, \end{aligned} \tag{3.2.3}$$

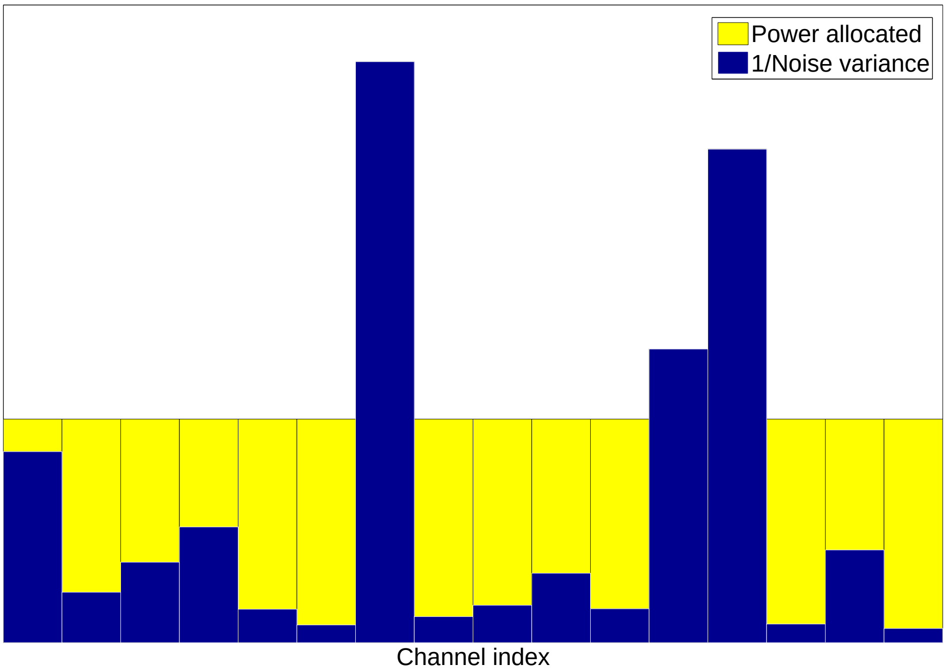


Figure 3.1: A typical waterfilling solution for parallel Gaussian channels showing the inverse noise variance levels and optimal power allocations for each channel.

is given by

$$p^*(k) = \left(\mu - \frac{1}{\sigma^2(k)}\right)^+, \tag{3.2.4}$$

where μ *is chosen suitably to satisfy* $\sum_{k=1}^{N} p(k) = P$*, with* $(x)^+ \triangleq \max(0, x)$*.*

A typical solution is illustrated in Figure 3.1. The reason for the term "waterfilling" is evident from the figure. If the inverses of the noise variances are assumed to be represented by the topography of the bottom of a vessel and the total power to be allocated by a certain amount of water, then the waterfilling solution indicates that the optimal power allocation across the channels is given by the depth of the water when the water is poured into the vessel. The water level is reflected in the parameter μ.

3.2.2 MIMO Gaussian channel

The Gaussian channel model can be extended to describe single-user systems with multiple transmitter and/or receiver antennas. Let n_T and n_R be the number of transmitter and receiver antennas respectively. The MIMO Gaussian channel is given by

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{3.2.5}$$

where $\mathbf{y} \in \mathbb{C}^{n_R \times 1}$ is the signal at the receiver, $\mathbf{x} \in \mathbb{C}^{n_T \times 1}$ is the transmitted signal, $\mathbf{H} \in \mathbb{C}^{n_R \times n_T}$ is the (nonsingular) channel matrix and $\mathbf{n} \in \mathbb{C}^{n_R \times 1}$ is the receiver noise, which is assumed to be a zero-mean circularly symmetric complex Gaussian vector with the covariance matrix $\mathbf{I}_{n_R}$.

The observed information rate for this system is given by [64]

$$R = \log\det\left(\mathbf{I}_{n_R} + \mathbf{H}\mathbf{Q}\mathbf{H}^H\right). \tag{3.2.6}$$

where $\mathbf{Q} \triangleq \mathrm{E}\{\mathbf{x}\mathbf{x}^H\}$ is the covariance matrix of the transmitted signal. If the total power available is P, the rate-maximizing power allocation is given by the following result [65]:

PROPOSITION 3.2. *The solution to the optimization problem*

$$\begin{aligned} \max_{\mathbf{Q}} \quad & \log\det\left(\mathbf{I}_{n_R} + \mathbf{H}\mathbf{Q}\mathbf{H}^H\right) \\ \text{s.t.} \quad & \mathbf{Q} \succeq 0, \\ & \mathrm{Tr}(\mathbf{Q}) = P, \end{aligned} \tag{3.2.7}$$

is given by

$$\mathbf{Q}^* = \mathbf{U}\left(\mu\mathbf{I} - \mathbf{D}^{-1}\right)^+ \mathbf{U}^H \tag{3.2.8}$$

where $\mathbf{U}$ *and* $\mathbf{D}$ *are calculated from the eigendecomposition*

$$\mathbf{U}\mathbf{D}\mathbf{U}^H \triangleq \mathbf{H}^H\mathbf{H}. \tag{3.2.9}$$

Thus, for MIMO Gaussian channels, the optimal power allocation is to perform waterfilling along "spatial modes" of the channels, as suggested by its eigendecomposition.

3.3 Gaussian interference channel — multi-user systems

In an environment with multiple users, the previous two models break down. This is because the channels of different users are no longer independent, and transmission by one user causes interference to the others. Such a scenario is modelled as a Gaussian interference channel (GIC). In this model, the interference caused by other channels is modelled as an additive Gaussian noise in addition to the usual receiver noise. When there are many parallel Gaussian interference channels, the resulting wideband system is called a *vector* Gaussian interference channel. The vector Gaussian interference channel could be *frequency-flat* or *frequency-selective* in nature.[4]

The capacity region of the Gaussian interference channel is an open problem in information theory [64] and has been an area of active research [66, 67, 68, 69, 70]. A myopic approach where each user independently and simultaneously performs classical waterfilling [64] based only on its own channel state information (CSI) without taking the interference caused by other users into account will be inefficient and unstable when the cross-channel gains are not small enough that the interference caused is negligible. The alternate approach of modelling the rate-maximization problem as a single sum-rate maximization problem involving all the users and computed by a centralized controller, leads to a non-convex optimization problem and has been shown to be strongly NP–hard [22].

The focus in this thesis is on systems where the various users act with minimal coordination, which precludes the use of multi-user coding/decoding and interference-cancelation techniques. This is of interest because practical systems have significant limitations on decoder complexity, signalling and coordination among users. Under these constraints, multi-user interference is treated as noise and the design of the transmission strategy reduces to finding the optimum power allocation for each user. Due to the nature

[4]In this thesis, it is assumed that the various users in the vector Gaussian interference channel (the frequency-selective Gaussian interference channel, in particular) are single-antenna (SISO) systems, unless specifically stated.

of competition and interdependent performance, game theory provides an excellent set of tools to analyze this problem.

3.3.1 Competitive rate-maximization in the Gaussian interference channel

Cooperative and noncooperative game theory have both been extensively applied to analyze power control problems in wireless networks [71,72,73,74,75]. Cooperative game-theoretic approaches to the problem of power control in wireless networks have been surveyed in [76, 77, 78]. The distributed power control problem for both single-channel and multi-channel wireless networks has been characterized using supermodular game theory in [79, 80]. Coalition, coordination and Nash bargaining theory for resource allocation in interference channels have be investigated in [81, 82, 83, 84, 85, 86].

In this thesis, the noncooperative scenario where the users are competing against one another and aiming to maximize their own information rates is considered. This approach transforms the centralized multi-objective optimization problem into a set of mutually coupled competitive single-objective optimization problems. This competitive rate-maximization problem can be modelled as a strategic noncooperative game. The Nash equilibrium [42] of this game can be achieved via a distributed waterfilling algorithm where each user performs waterfilling by considering the multi-user interference as an additive coloured noise.

The seminal work on competitive rate-maximization [24] has used a game-theoretic approach to design a decentralized algorithm for two-user dynamic power control in a digital subscriber line (DSL) environment modelled as a frequency-selective Gaussian interference channel. This work has proposed a sequential iterative waterfilling algorithm for reaching the Nash equilibrium in a distributed manner. A Nash equilibrium of the rate-maximization game implies that given that the power allocations of other users is constant, no user can further increase the achieved information rate unilaterally. A vector power control problem for frequency-flat Gaussian interference channels has been presented in [87]. The issue of multiple Nash equilibria occurring in the presence of strong interference has been investigated in [88]. Analysis of the sequential iterative waterfilling algorithm for an arbitrary number of users using a linear complementary problem formulation has been presented in [89]. Sufficient conditions for global convergence of an asynchronous iterative waterfilling algorithm by formulating the waterfilling function as a piecewise affine function have been presented in [90]. A matrix game formulation for competitive rate-maximization in frequency-selective Gaussian interference channels, along with a novel interpretation of the waterfilling function as an Euclidian projection of a vector onto a convex set have been presented in [91,92,93]. The convergence properties of the iterative waterfilling algorithm to multiple Nash equilibria in flat-fading Gaussian interference channels under different levels of interference and different update strategies have been investigated in [94, 95, 96] and with sequential update strategy in frequency-selective Gaussian interference channels in [97].

An early work, [98], has looked into waterfilling algorithms for MIMO interference systems, but without any analytical results on the existence of an equilibrium and global convergence of the algorithm. The different approaches in [89, 90, 93] for the iterative waterfilling algorithm in frequency-selective Gaussian interference channels have been unified under the formulation of the waterfilling function as a Euclidean projection and extended to MIMO Gaussian interference channels with square (nonsingular) channels in [63]. A complete characterization of the MIMO rate-maximization game and the MIMO waterfilling algorithm for arbitrary channel matrices has been investigated in [99]. This has been extended to the MIMO cognitive radio scenario in [100].

The subsequent sections present an overview of the rate-maximization game for frequency-selective and MIMO Gaussian interference channels, along with the sufficient conditions for the existence and uniqueness of the Nash equilibrium and convergence of the iterative waterfilling algorithm to the equilibrium. The Euclidean projection interpretations of the respective waterfilling functions are also presented as they are useful in the analysis of the properties of the equilibrium.

3.4 Iterative waterfilling for frequency-selective GICs

3.4.1 System model

Consider a Gaussian frequency-selective interference channel with N frequencies, composed of Q SISO links. The quantity $H_{qr}(k)$ denotes the frequency response for the kth frequency bin of the channel between source r and destination q. The variance of the zero-mean circularly symmetric complex Gaussian noise at receiver q over the frequency bin k is denoted by $\sigma_q^2(k)$. The channel is assumed to be quasi-stationary for the duration of the transmission. Each receiver is assumed to know the channel between itself and the corresponding transmitter, but not other transmitters. Also, each receiver is assumed to be able to measure, with no errors, the overall power spectral density of the noise plus multi-user interference generated by other users. Based on this information, each receiver computes the optimal power allocation across the frequency bins for its own link and transmits it back to the corresponding transmitter through a low bit-rate error-free feedback channel. Let the vector $\mathbf{s}_q \triangleq [s_q(1)s_q(2)\dots s_q(N)]$ be the N symbols transmitted by user q on the N frequency bins and $p_q(k) \triangleq E\{|s_q(k)|^2\}$ be the power allocated to the kth frequency bin by user q and $\mathbf{p}_q \triangleq [p_q(1)p_q(2)\dots p_q(N)]$ be the power allocation vector. The maximum achievable information rate by user q is given by [64]

$$R_q = \frac{1}{N}\sum_{k=1}^{N} \log(1 + \mathrm{sinr_q}(k)), \tag{3.4.1}$$

where $\mathrm{sinr_q}(k)$ is the signal-to-interference plus noise ratio (SINR) on the kth frequency bin of the qth user,

$$\mathrm{sinr_q}(k) \triangleq \frac{|H_{qq}(k)|^2 p_q(k)}{\sigma_q^2(k) + \sum_{r \neq q} |H_{qr}(k)|^2 p_r(k)}. \tag{3.4.2}$$

3.4.2 Rate-maximization game

The problem of power allocation across the frequency bins is cast as strategic noncooperative game with the SISO links as players and their information rates as pay-off functions, under the following two constraints:

- Maximum total transmit power for each user:

  $$\mathrm{E}\left\{\|\mathbf{s}_q\|_2^2\right\} = \sum_{k=1}^{N} p_q(k) \leq N P_q, \qquad q = 1, \ldots, Q, \tag{3.4.3}$$

  where P_q is power in units of energy per transmitted symbol.

- Spectral mask constraints:

  $$\mathrm{E}\left\{|s_q(k)|^2\right\} = p_q(k) \leq p_q^{max}(k) \tag{3.4.4}$$

  for $k = 1, \ldots, N$ and $q = 1, \ldots, Q$, where $p_q^{max}(k)$ is the maximum power that is allowed to be allocated by user q for the frequency bin k.

Mathematically, the game can be written as

$$(\mathscr{G}^S): \quad \begin{array}{ll} \max\limits_{\mathbf{p}_q} & \dfrac{1}{N}\sum\limits_{k=1}^{N}\log(1+\mathrm{sinr_q}(k)) \\ \text{s.t.} & \mathbf{p}_q \in \mathcal{P}_q \end{array} \qquad \forall q \in \Omega \tag{3.4.5}$$

where $\Omega \triangleq \{1, \dots, Q\}$ is the set of the Q players (i.e. the SISO links) and $\mathcal{P}_q$ is the set of admissible strategies of user q, which is defined as

$$\mathcal{P}_q \triangleq \left\{\mathbf{p}_q \in \mathbb{R}^N : \frac{1}{N}\sum_{k=1}^{N} p_q(k) = 1,\ 0 \leq p_q(k) \leq p_q^{max}(k),\ k = 1, \dots, N\right\}. \tag{3.4.6}$$

The inequality constraint in (4.2.1) is replaced with the equality constraint as, at the optimum of each problem in (3.4.5), the constraint must be satisfied with equality. To avoid the trivial solution $p_q(k) = p_q^{max}(k)\ \forall k$, it is assumed that $\sum_{k=1}^{N} p_q^{max} > N$. Further, the players can be limited to pure strategies instead of mixed strategies, as shown in [91].

3.4.3 Nash equilibrium

The solution to the game $\mathscr{G}^S$ is the Nash equilibrium. At any Nash equilibrium of this game, the optimum action profile of the players $\{\mathbf{p}_q^\star\}_{q\in\Omega}$ must satisfy the following set of nonlinear equations:

$$\mathbf{p}_q^\star = \mathrm{WF_q^S}(\mathbf{p}_1^\star, \dots, \mathbf{p}_{q-1}^\star, \mathbf{p}_{q+1}^\star, \dots, \mathbf{p}_Q^\star) = \mathrm{WF_q^S}(\mathbf{p}_{-q}^\star) \quad \forall q \in \Omega. \tag{3.4.7}$$

The waterfilling operator $\mathrm{WF}_{\mathrm{q}}^{\mathrm{S}}(\cdot)$ is defined as [93]

$$\left[\,\mathrm{WF}_{\mathrm{q}}^{\mathrm{S}}(\mathbf{p}_{-q})\right]_k \triangleq \left[\mu_q^S - \frac{\sigma_q^2(k) + \sum_{r\neq q} |H_{qr}(k)|^2 p_r(k)}{|H_{qq}(k)|^2}\right]_0^{p_q^{max}(k)} \qquad k = 1,\ldots,N \tag{3.4.8}$$

where μ_q^S is chosen to satisfy the power constraint $\frac{1}{N}\sum_{k=1}^{N} p_q^* = 1$.The Euclidean projection $[x]_a^b$ is defined as

$$[x]_a^b = \begin{cases} a & \text{if } x \leq a \\ x & \text{if } a < x < b \\ b & \text{if } x \geq b \end{cases} \tag{3.4.9}$$

Waterfilling as a projection

The waterfilling function $\mathrm{WF}_{\mathrm{q}}^{\mathrm{S}}(\cdot)$ can be equivalently represented as a Euclidean projection. This is the key concept that allows the analytical study of convergence of the iterative waterfilling-based algorithms.

THEOREM 3.3. *The waterfilling operation* $\mathrm{WF}_{\mathrm{q}}^{\mathrm{S}}(\cdot)$ *in* (3.4.8) *can be equivalently written as [92]*

$$\mathrm{WF}_{\mathrm{q}}^{\mathrm{S}}(\mathbf{p}_{-q}) = \left[-\,\mathbf{isnr}_{\mathrm{q}}(\mathbf{p}_{-q})\right]_{\mathcal{P}_q}, \tag{3.4.10}$$

where

$$\left[\mathbf{isnr}_{\mathrm{q}}(\mathbf{p}_{-q})\right]_k \triangleq \frac{\sigma_q^2(k) + \sum_{r\neq q} |H_{qr}(k)|^2 p_r(k)}{|H_{qq}(k)|^2}. \tag{3.4.11}$$

Existence and uniqueness of the Nash equilibrium

Let $\mathcal{D}_q^{min} \subseteq \{1, \ldots, N\}$ denote the set of frequency bins $\{1, \ldots, N\}$ from which the frequency bins that user q would never use as the best response set to any strategies adopted by the other users are removed [91]

$$\mathcal{D}_q^{min} \triangleq \Big\{ k \in \{1, \ldots, N\} : \; \exists \, \mathbf{p}_{-q} \in \mathcal{P}_{-q} \text{ such that } \left[\mathrm{WF}_{\mathrm{q}}^{\mathrm{S}}(\mathbf{p}_{-q}) \right]_k \neq 0 \Big\} \tag{3.4.12}$$

where $\mathcal{P}_{-q} \triangleq \mathcal{P}_1 \times \cdots \times \mathcal{P}_{q-1} \times \mathcal{P}_{q+1} \times \cdots \times \mathcal{P}_Q$. Given the game $\mathscr{G}^S$, the non-negative matrix $\mathbf{S}^{max} \in \mathbb{R}_+^{Q \times Q}$ is defined as

$$[\mathbf{S}^{max}]_{qr} \triangleq \begin{cases} \max\limits_{k \in \mathcal{D}_q \cap \mathcal{D}_r} \dfrac{|H_{qr}(k)|^2}{|H_{qq}(k)|^2} \dfrac{P_r}{P_q}, & \text{if } r \neq q, \\ 0, & \text{otherwise} \end{cases} \tag{3.4.13}$$

The sufficient condition for existence and uniqueness of the Nash equilibrium of game $\mathscr{G}^S$ is given by the following theorem [91]:

THEOREM 3.4. *Game $\mathscr{G}^S$ has at least one Nash equilibrium for any set of channel matrices and transmit powers of the users. Furthermore, the Nash equilibrium is unique if*

$$\rho(\mathbf{S}^{max}) < 1 \tag{3.4.14}$$

where $\mathbf{S}^{max}$ is defined in (3.4.13).

3.4.4 Asynchronous iterative waterfilling algorithm

Let the discrete set $\mathbb{T} \subseteq \mathbb{N}_+ = 1, 2, \ldots$ be the set of times at which one or more users update their strategies. Let $\mathbf{p}_q^{(n)}$ denote the vector power allocation of user q at the discrete time n, and let $\mathbb{T}_q \subseteq \mathbb{T}$ represent the set of time instants n when the power vector $\mathbf{p}_q^{(n)}$ of user q is updated. Let $\tau_r^q(n)$ denote the time when the most recently perceived interference from user r was computed by user q at time n (Note that $0 \leq \tau_r^q(n) \leq n$). Hence, if user q updates his strategy at time n, then

$$\mathbf{p}_{-q}^{\left(\tau^q(n)\right)} \triangleq \left(\mathbf{p}_1^{\left(\tau_1^q(n)\right)}, \ldots, \mathbf{p}_{q-1}^{\left(\tau_{q-1}^q(n)\right)}, \mathbf{p}_{q+1}^{\left(\tau_{q+1}^q(n)\right)}, \ldots, \mathbf{p}_Q^{\left(\tau_Q^q(n)\right)}\right). \tag{3.4.15}$$

The asynchronous iterative waterfilling algorithm (AIWFA) for computing the Nash equilibrium of game $\mathscr{G}^S$ in a distributed fashion is described in Algorithm 3.1. The convergence of Algorithm 3.1 is guaranteed under the following sufficiency condition [93]:

THEOREM 3.5. *The asynchronous iterative waterfilling algorithm described in Algorithm 3.1 converges to the unique Nash equilibrium of game $\mathscr{G}^S$ as $T \rightarrow \infty$ for any set of feasible initial conditions if condition* (3.4.14) *is satisfied.*

The global convergence of the asynchronous iterative waterfilling algorithm to the unique Nash equilibrium is guaranteed by Theorem 3.5 using condition (3.4.13) despite game $\mathscr{G}^S$ and the waterfilling operation $\mathrm{WF}_{\mathrm{q}}^{\mathrm{S}}(\cdot)$ being nonlinear.

Algorithm 3.1 – Asynchronous Iterative Waterfilling Algorithm

Input:
Ω: Set of users in the system
$\mathcal{P}_q$: Set of admissible strategies of user q
$\mathbb{T}_q$: Set of time instants n when the power vector $\mathbf{p}_q^{(n)}$ of user q is updated
T: Number of iterations for which the algorithm is run
$\tau_r^q(n)$: Time of the most recent power allocation of user r available to user q at time n
$\mathrm{WF}_{\mathrm{q}}^{\mathrm{S}}(\cdot)$: Waterfilling operation in (3.4.8)
Initialization: $n = 0$ and $\mathbf{p}_q^{(0)} \leftarrow$ any $\mathbf{p} \in \mathcal{P}_q,\ \forall q \in \Omega$
for $n = 0$ to T **do**

$$\mathbf{p}_q^{(n+1)} = \begin{cases} \mathrm{WF}_{\mathrm{q}}^{\mathrm{S}}\left(\mathbf{p}_{-q}^{(\tau^q(n))}\right), & \text{if } n \in \mathbb{T}_q, \\ \mathbf{p}_q^{(n)}, & \text{otherwise,} \end{cases} \quad \forall q \in \Omega.$$

end for

3.5 Iterative waterfilling for MIMO GICs

3.5.1 System model

Consider a narrowband MIMO Gaussian interference channel composed of Q MIMO links. The signal vector $\mathbf{y}_q \in \mathbb{C}^{n_{R_q} \times 1}$ measured at the receiver of user q is

$$\mathbf{y}_q = \mathbf{H}_{qq}\mathbf{x}_q + \sum_{r \neq q} \mathbf{H}_{rq}\mathbf{x}_r + \mathbf{n}_q \tag{3.5.1}$$

where $\mathbf{H}_{rq} \in \mathbb{C}^{n_{R_q} \times n_{T_r}}$ is the channel matrix between source r and destination q, $\mathbf{x}_q \in \mathbb{C}^{n_{T_q} \times 1}$ is the signal vector transmitted by source q and $\mathbf{n}_q \in \mathbb{C}^{n_{R_q} \times 1}$ is the receiver noise vector of user q, which is assumed to be a zero-mean circularly symmetric complex Gaussian vector with an arbitrary (nonsingular) covariance matrix $\mathbf{R}_{n_q}$. The second term in the right hand side of (3.5.1) is the multi-user interference observed at the destination q, which is treated as additive spatially coloured Gaussian noise at the receiver of user q. The channel is assumed to be quasi-stationary for the duration of the transmission. At each receiver q, the channel matrix $\mathbf{H}_{qq}$ is assumed to be known. Also, each receiver is assumed to be able to measure the covariance matrix of the noise plus multi-user interference generated by other users. Based on this information, each destination q computes the optimal covariance matrices $\mathbf{Q}_q \triangleq \mathrm{E}\{\mathbf{x}_q {\mathbf{x}_q}^H\}$ for its own link and transmits it back to its transmitter through a low bit-rate error-free feedback channel. This gives the optimal transmitter beamformer for each of the users. The information rate of user q, $R_q(\mathbf{Q}_q, \mathbf{Q}_{-q})$, for this system can be written as [63]

$$R_q(\mathbf{Q}_q, \mathbf{Q}_{-q}) = \log\det(\mathbf{I} + \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\mathbf{Q}_q) \qquad (3.5.2)$$

where

$$\mathbf{R}_{-q}(\mathbf{Q}_{-q}) \triangleq \mathbf{R}_{\mathbf{n}_q} + \sum_{r \neq q} \mathbf{H}_{rq}\mathbf{Q}_r\mathbf{H}_{rq}^H \qquad (3.5.3)$$

is the interference plus noise covariance matrix observed by destination q, and $\mathbf{Q}_{-q} \triangleq \{\mathbf{Q}_r\}_{r \neq q}^{Q}$ is the set of covariance matrices of all users except the qth user.

3.5.2 Rate-maximization game

Consider the system in (3.5.1) as a strategic noncooperative game with the MIMO links as players and information rates of the respective links as payoff functions. Each player q competes rationally against other users in order to maximize its own information rate $R_q(\mathbf{Q}_q, \mathbf{Q}_{-q})$ by designing the optimal covariance matrix $\mathbf{Q}_q^\star$, given the constraint

$$\mathrm{E}\{\|\mathbf{x}_q\|_2^2\} = \mathrm{Tr}(\mathbf{Q}_q) \leq P_q \tag{3.5.4}$$

where P_q is the maximum average power transmitted in units of energy per transmission. Mathematically, the game can be written as

$$(\mathscr{G}^M) : \quad \begin{array}{ll} \max\limits_{\mathbf{Q}_q} & R_q(\mathbf{Q}_q, \mathbf{Q}_{-q}) \\ \mathrm{s.\,t.} & \mathbf{Q}_q \in \mathscr{Q}_q \end{array} \qquad \forall q \in \Omega \tag{3.5.5}$$

where $\Omega \triangleq \{1, \ldots, Q\}$ is the set of the Q players (i.e. MIMO links), $R_q(\mathbf{Q}_q, \mathbf{Q}_{-q})$ is the payoff function of player q as given in (3.5.2) and the set of admissible strategies of player q, $\mathscr{Q}_q$, is defined as

$$\mathscr{Q}_q \triangleq \{\mathbf{Q} \in \mathbb{C}^{n_{T_q} \times n_{T_q}} : \mathbf{Q} \succeq \mathbf{0}, \quad \mathrm{Tr}(\mathbf{Q}_q) = P_q\}. \tag{3.5.6}$$

The inequality constraint in (3.5.4) is replaced with the equality constraint as, at the optimum of each problem in (3.5.5), the constraint must be satisfied with equality [99]. Further, it can be proved that the players can be limited to pure strategies instead of mixed strategies, as shown in [91].

3.5.3 Nash equilibrium

The solution to the game $\mathscr{G}^M$ is the Nash equilibrium. Given $\mathbf{Q}_{-q} \in \mathscr{Q}_{-q} \triangleq \mathscr{Q}_1 \times \cdots \times \mathscr{Q}_{q-1} \times \mathscr{Q}_{q+1} \times \cdots \mathscr{Q}_Q$ at any Nash equilibrium of this game, the optimum action profile of the players $\{\mathbf{Q}_q^\star\}_{q\in\Omega}$ must satisfy

$$\mathbf{Q}_q^\star = \mathrm{WF}_\mathrm{q}^\mathrm{M}(\mathbf{Q}_1^\star, \ldots, \mathbf{Q}_{q-1}^\star, \mathbf{Q}_{q+1}^\star, \ldots, \mathbf{Q}_Q^\star) = \mathrm{WF}_\mathrm{q}^\mathrm{M}(\mathbf{Q}_{-q}^\star) \quad \forall q \in \Omega. \tag{3.5.7}$$

The waterfilling operator $\mathrm{WF}_\mathrm{q}^\mathrm{M}(\cdot)$ is defined as [99]

$$\mathrm{WF}_\mathrm{q}^\mathrm{M}(\mathbf{Q}_{-q}) \triangleq \mathbf{U}_q(\mu_q^M \mathbf{I} - \mathbf{D}_q^{-1})^+ \mathbf{U}_q^H \tag{3.5.8}$$

where μ_q^M is chosen to satisfy $\mathrm{Tr}\left((\mu_q^M \mathbf{I} - \mathbf{D}_q^{-1})^+\right) = P_q$. The (semi)-unitary matrix of eigenvectors $\mathbf{U}_q = \mathbf{U}_q(\mathbf{Q_{-q}}) \in \mathbb{C}^{n_{T_q} \times r_q}$ and the diagonal matrix $\mathbf{D}_q = \mathbf{D}_q(\mathbf{Q}_{-q}) \in \mathbb{R}_{++}^{r_q \times r_q}$ with $r_q \triangleq \mathrm{rank}(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}) = \mathrm{rank}(\mathbf{H}_{qq})$ positive eigenvalues are calculated from the eigendecomposition

$$\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq} \triangleq \mathbf{U}_q \mathbf{D}_q \mathbf{U}_q^H. \tag{3.5.9}$$

Waterfilling operation as a projection

The waterfilling operation $\mathrm{WF}_\mathrm{q}^\mathrm{M}(\cdot)$ can be interpreted as a matrix projection onto a convex set [92]. The projection expression depends on the nature of the channel matrices $\mathbf{H}_{qq}$, i.e. whether it is square and nonsingular or not.

THEOREM 3.6. *For the system in* (3.5.1) *with an arbitrary (possibly singular) set of channel matrices, the MIMO waterfilling operator* $\mathrm{WF}_\mathrm{q}^\mathrm{M}(\mathbf{Q}_{-q})$ *in*

(3.5.8) *can be equivalently written as [99]*

$$\mathrm{WF_q^M}(\mathbf{Q}_{-q}) = \left[-\left((\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq})^{\sharp} + c_q \mathbf{P}_{\mathcal{N}(\mathbf{H}_{qq})}\right)\right]_{\mathscr{Q}_q} \tag{3.5.10}$$

where $c_q \geq c_q(\mathbf{Q}_{-q}) \triangleq P_q + \max_{i\in 1,\ldots,r_q} \left[\mathbf{D}_q(\mathbf{Q}_{-q})\right]_{ii}^{-1}$ *is a positive constant.*

The projection operator $\mathbf{P}_{\mathcal{N}(\mathbf{A})}$ is defined as

$$\mathbf{P}_{\mathcal{N}(\mathbf{A})} \triangleq \mathbf{N_A}({\mathbf{N_A}}^H \mathbf{N_A})^{-1}{\mathbf{N_A}}^H \tag{3.5.11}$$

where $\mathbf{N_A}$ is any matrix whose columns span the null space of $\mathbf{A}$ given by $\mathcal{N}(\mathbf{A})$.

COROLLARY 3.6.1. *If the direct channel matrices* $\mathbf{H}_{qq}$ *are square and nonsingular for every user q, then the MIMO waterfilling operator* $\mathrm{WF_q^M}(\mathbf{Q}_{-q})$ *in* (3.5.8) *can be equivalently written as [63]*

$$\mathrm{WF_q^M}(\mathbf{Q}_{-q}) = \left[-\left(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\right)^{-1}\right]_{\mathscr{Q}_q} \tag{3.5.12}$$

In this case, $\mathbf{U}_q$ becomes a unitary matrix and $\mathbf{D}_q$ will be of dimension $n_{T_q} \times n_{T_q}$ with n_{T_q} eigenvalues as the matrix $\mathbf{H}_{qq}$ will be a full-column rank matrix with $r_q = \mathrm{rank}(\mathbf{H}_{qq}) = n_{T_q}$. It can easily be verified that the result in (3.5.12) is a special case of (3.5.10), as $\mathbf{P}_{\mathcal{N}(\mathbf{H}_{qq})} = 0$ when the channel matrices $\mathbf{H}_{qq}$ are square and nonsingular, and thus full column-rank.

Existence and uniqueness of the Nash equilibrium

Given the MIMO system in (3.5.1), the non-negative matrices $\mathbf{S} \in \mathbb{R}_+^{Q\times Q}$ and $\mathbf{S}^{up} \in \mathbb{R}_+^{Q\times Q}$ are defined as

$$[\mathbf{S}]_{qr} \triangleq \begin{cases} \rho(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{\sharp H} \mathbf{H}_{qq}^{\sharp} \mathbf{H}_{rq}), & \text{if } r \neq q, \\ 0, & \text{otherwise} \end{cases} \tag{3.5.13}$$

and

$$[\mathbf{S}^{\mathrm{up}}]_{qr} \triangleq \begin{cases} \mathrm{innr_q} \cdot \rho(\mathbf{H}_{rq}^H \mathbf{H}_{rq}) \rho(\mathbf{H}_{qq}^{\sharp H} \mathbf{H}_{qq}^{\sharp}), & \text{if } r \neq q, \\ 0, & \text{otherwise,} \end{cases} \tag{3.5.14}$$

where $\mathrm{innr_q}$ is the interference-plus-noise to noise ratio and is defined as [99]

$$\mathrm{innr_q} \triangleq \frac{\rho\left(\mathbf{R}_{n_q} + \sum_{r\neq q} P_r \mathbf{H}_{rq} \mathbf{H}_r q^H\right)}{\lambda_{\min}(\mathbf{R}_{n_q})} \geq 1, \quad \forall q \in \Omega. \tag{3.5.15}$$

Given $\mathbf{S}^{\mathrm{up}}$ and $\mathbf{S}$, the matrix $\overline{\mathbf{S}}^{\mathrm{up}} \in \mathbb{R}_+^{Q\times Q}$ is defined as

$$[\overline{\mathbf{S}}^{\mathrm{up}}]_{qr} \triangleq \begin{cases} [\mathbf{S}]_{qr}, & \text{if } \mathrm{rank}(\mathbf{H}_{qq}) = n_{R_q}, \\ [\mathbf{S}^{\mathrm{up}}]_{qr}, & \text{otherwise,} \end{cases} \tag{3.5.16}$$

The sufficient condition for existence and uniqueness of the Nash equilibrium of game $\mathscr{G}^M$ is given by the following theorem [99]:

THEOREM 3.7. *Game $\mathscr{G}^M$ has at least one Nash equilibrium for any set of channel matrices and transmit powers of the users. Furthermore, the Nash* *equilibrium is unique if*

$$\rho(\overline{\mathbf{S}}^{\mathrm{up}}) < 1 \qquad (3.5.17)$$

where $\overline{\mathbf{S}}^{\mathrm{up}}$ *is defined in* (3.5.16)

COROLLARY 3.7.1. *If the direct channel matrices* $\mathbf{H}_{qq}$ *are square and non-singular for every user* q*, then the sufficient condition for the uniqueness of the Nash equilibrium of game* $\mathscr{G}^M$ *is [63]*

$$\rho(\overline{\mathbf{S}}) < 1 \qquad (3.5.18)$$

where $\overline{\mathbf{S}}$ *is defined as*

$$[\overline{\mathbf{S}}]_{qr} \triangleq \begin{cases} \rho(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{-H} \mathbf{H}_{qq}^{-1} \mathbf{H}_{rq}), & \text{if } r \neq q, \\ 0, & \text{otherwise.} \end{cases} \qquad (3.5.19)$$

3.5.4 MIMO iterative waterfilling algorithm

Let the discrete set $\mathbb{T} \subseteq \mathbb{N}_+ = 1, 2, \ldots$ be the set of times at which one or more users update their strategies. Let $\mathbf{Q}_q^{(n)}$ denote the covariance matrix of user q at the discrete time n, and let $\mathbb{T}_q \subseteq \mathbb{T}$ denote the set of time instants n when the covariance matrix $\mathbf{Q}_q^{(n)}$ of user q is updated. Let $\tau_r^q(n)$ denote the time when the most recently perceived interference from user r was computed by user q at time n (Note that $0 \leq \tau_r^q(n) \leq n$). Hence, if user

Algorithm 3.2 – MIMO Iterative Waterfilling Algorithm

Input:

Ω: Set of users in the system

$\mathscr{Q}_q$: Set of admissible strategies of user q

$\mathbb{T}_q$: Set of time instants n when the covariance matrix $\mathbf{Q}_q^{(n)}$ of user q is updated

T: Number of iterations for which the algorithm is run

$\tau_r^q(n)$: Time of the most recent power allocation of user r available to user q at time n

$\mathrm{WF}_{\mathrm{q}}^{\mathrm{M}}(\cdot)$: Waterfilling operation in (3.5.8)

Initialization: $n = 0$ and $\mathbf{Q}_q^{(0)} \leftarrow$ any $\mathbf{Q} \in \mathscr{Q}_q, \ \forall q \in \Omega$

for $n = 0$ to T **do**

$$\mathbf{Q}_q^{(n+1)} = \begin{cases} \mathrm{WF}_{\mathrm{q}}^{\mathrm{M}}\left(\mathbf{Q}_{-q}^{(\tau^q(n))}\right), & \text{if}(n) \in \mathbb{T}_q, \\ \mathbf{Q}_q^{(n)}, & \text{otherwise} \end{cases} \qquad \forall q \in \Omega,$$

end for

q updates his strategy at time n, then

$$\mathbf{Q}_{-q}^{\left(\tau^q(n)\right)} \triangleq \left(\mathbf{Q}_1^{\left(\tau_1^q(n)\right)}, \ldots, \mathbf{Q}_{q-1}^{\left(\tau_{q-1}^q(n)\right)}, \mathbf{Q}_{q+1}^{\left(\tau_{q+1}^q(n)\right)}, \ldots, \mathbf{Q}_Q^{\left(\tau_Q^q(n)\right)}\right). \quad (3.5.20)$$

The MIMO iterative waterfilling algorithm (MIWFA) for computing the Nash equilibrium of game $\mathscr{G}^M$ in a distributed fashion is described in Algorithm 3.2. The convergence of Algorithm 3.2 is guaranteed under the following sufficiency condition [99]:

Theorem 3.8. *The MIMO iterative waterfilling algorithm, described in Algorithm 3.2 converges to the unique Nash equilibrium of game $\mathscr{G}^M$ as $T \to \infty$ for any set of feasible initial conditions if condition* (3.5.17) *is satisfied.*

The global convergence of the MIMO iterative waterfilling algorithm to the unique Nash equilibrium is guaranteed by Theorem 3.8 using condition (3.5.17) despite game $\mathscr{G}^M$ and the waterfilling operation $\mathrm{WF}_{\mathrm{q}}^{\mathrm{M}}(\cdot)$ being non-linear and is valid for arbitrary channel matrices, either tall/fat or singular.

3.6 Iterative waterfilling for broadband MIMO GICs

The previous section introduced the MIMO iterative waterfilling algorithm for narrowband systems. Here, this framework is extended to broadband (OFDM) systems, limited to systems with square (nonsingular) direct channel matrices. The challenge in such a scenario is that the power allocation has to be performed across both frequency and space (viz. transmit antennas). This is achieved by modifying the constraints in the waterfilling expression so that power allocation is optimal in both space and frequency. The spatial optimality is essentially designing the optimal beamformer and the spectral optimality is the magnitude of power spectral density at each frequency bin.

3.6.1 System model

Consider a broadband MIMO Gaussian interference channel with N frequencies composed of Q MIMO links. At any frequency f, the signal vector $\mathbf{y}_q^f \in \mathbb{C}^{n_{R_q} \times 1}$ measured at the receiver of user q is

$$\mathbf{y}_q^f = \mathbf{H}_{qq}^f \mathbf{x}_q^f + \sum_{r \neq q} \mathbf{H}_{rq}^f \mathbf{x}_r^f + \mathbf{n}_q^f, \tag{3.6.1}$$

where $\mathbf{H}_{rq}^f \in \mathbb{C}^{n_{R_q} \times n_{T_r}}$ is the channel matrix between source r and destination q, $\mathbf{x}_q^f \in \mathbb{C}^{n_{T_q} \times 1}$ is the vector transmitted by source q and $\mathbf{n}_q^f \in \mathbb{C}^{n_{R_q} \times 1}$ is the receiver noise of user q, which is a zero-mean circularly symmetric complex Gaussian vector with an arbitrary (nonsingular) covariance matrix $\mathbf{R}_{n_q}^f$. The second term in the right hand side of (3.6.1) is the multi-user interference observed at the destination q, which is treated as additive spatially coloured noise. The channel is assumed to be stationary for the duration of the transmission. At each receiver q, the channel matrix $\mathbf{H}_{qq}^f$ is assumed to be known. Note that there are no constraints on the dimensions or rank of the channel matrices. Also, each receiver is assumed to be able to measure the covariance matrix of the noise plus multi-user interference generated by other users. The covariance matrix of the noise plus multi-user interference observed by destination q at frequency f is given by

$$\mathbf{R}_{-q}^f \triangleq \mathbf{R}_{\mathbf{n}_q}^f + \sum_{r \neq q} \mathbf{H}_{rq}^f \mathbf{Q}_r^f {\mathbf{H}_{rq}^f}^H. \tag{3.6.2}$$

Based on this information, each destination q computes the optimal covariance matrices $\mathbf{Q}_q^f \triangleq \mathrm{E}\{\mathbf{x}_q^f {\mathbf{x}_q^f}^H\}$ for each frequency f for its own link and informs its transmitter through a low bit-rate error-free feedback channel.

Let $\widehat{\mathbf{Q}}_q$ and $\widehat{\mathbf{R}}_{n_q}$ be the set of the transmitter covariance matrices and noise covariance of player q for the N frequency bins written in block-diagonal form respectively,

$$\widehat{\mathbf{Q}}_q \triangleq \text{Diag}\left(\mathbf{Q}_q^1, \ldots, \mathbf{Q}_q^N\right) = \begin{bmatrix} \mathbf{Q}_q^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Q}_q^N \end{bmatrix}, \tag{3.6.3}$$

and

$$\widehat{\mathbf{R}}_{n_q} \triangleq \text{Diag}\left(\mathbf{R}_{n_q}^1, \ldots, \mathbf{R}_{n_q}^N\right) = \begin{bmatrix} \mathbf{R}_{n_q}^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{R}_{n_q}^N \end{bmatrix}. \tag{3.6.4}$$

The channel matrices and the noise plus multi-user interference covariance matrices can be written in block-diagonal form respectively as

$$\widehat{\mathbf{H}}_{rq} \triangleq \text{Diag}\left(\mathbf{H}_{rq}^1, \ldots, \mathbf{H}_{rq}^N\right) = \begin{bmatrix} \mathbf{H}_{rq}^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{H}_{rq}^N \end{bmatrix}, \tag{3.6.5}$$

and

$$\widehat{\mathbf{R}}_{-q} \triangleq \text{Diag}\left(\mathbf{R}_{-q}^1, \ldots, \mathbf{R}_{-q}^N\right) = \begin{bmatrix} \mathbf{R}_{-q}^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{R}_{-q}^N \end{bmatrix}. \tag{3.6.6}$$

The information rate achieved by user q over all frequency bins is given by

$$R_q(\widehat{\mathbf{Q}}_q, \widehat{\mathbf{Q}}_{-q}) = \log\det(\mathbf{I} + \widehat{\mathbf{H}}_{qq}^H \widehat{\mathbf{R}}_{-q}^{-1} \widehat{\mathbf{H}}_{qq} \widehat{\mathbf{Q}}_q). \tag{3.6.7}$$

Each player q competes rationally against other users in order to maximize its own information rate $R_q(\widehat{\mathbf{Q}}_q, \widehat{\mathbf{Q}}_{-q})$ by designing the optimal covariance matrix $\widehat{\mathbf{Q}}_q^\star$, given the constraint

$$\mathrm{E}\Big\{\sum_{f=1}^{N} \|\mathbf{x}_q^f\|_2^2\Big\} = \sum_{f=1}^{N} \mathrm{Tr}(\mathbf{Q}_q^f) = \mathrm{Tr}(\widehat{\mathbf{Q}}_q) \leq P_q, \tag{3.6.8}$$

where P_q is the maximum average power transmitted in units of energy per transmission.

3.6.2 Rate-maximization game

The game can be cast in mathematical form as

$$(\mathscr{G}^{BM}): \quad \begin{array}{ll} \max\limits_{\widehat{\mathbf{Q}}_q} & R_q(\widehat{\mathbf{Q}}_q, \widehat{\mathbf{Q}}_{-q}) \\ \text{s.t.} & \widehat{\mathbf{Q}}_q \in \widehat{\mathscr{Q}}_q, \end{array} \qquad \forall q \in \Omega, \tag{3.6.9}$$

where $\Omega \triangleq \{1, \ldots, Q\}$ is the set of the Q players (i.e. MIMO links), $R_q(\widehat{\mathbf{Q}}_q, \widehat{\mathbf{Q}}_{-q})$ is the payoff function of player q as given in (3.6.7) and the set of admissible strategies of player q, $\widehat{\mathscr{Q}}_q$, is defined as

$$\begin{aligned} \widehat{\mathscr{Q}}_q \triangleq \Big\{\widehat{\mathbf{Q}} : \sum_{f=1}^{N} \mathrm{Tr}(\mathbf{Q}_q^f) = P_q;\ \mathbf{Q}_q^f \in \mathbb{C}^{n_{T_q} \times n_{T_q}}, \\ \mathbf{Q}_q^f \succeq \mathbf{0} \quad \forall f = 1, \ldots, N\Big\}. \end{aligned} \tag{3.6.10}$$

This game can now be solved using the MIMO iterative waterfilling algorithm [99] as the new matrices $\widehat{\mathbf{Q}}_q, \widehat{\mathbf{R}}_{-q}$ and $\widehat{\mathbf{H}}_{rq}$ satisfy the necessary conditions and assumptions.

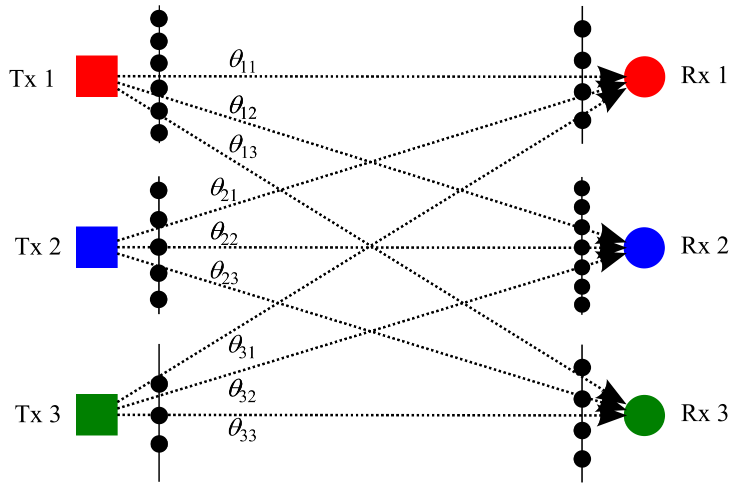


Figure 3.2: System with 3 MIMO links and 4 frequencies. θ_{qr} is the electrical angle of the signal observed at destination r from source q.

3.6.3 Numerical results

To confirm the operation of the algorithm in a broadband setup, a broadband system having 3 MIMO links and 4 frequencies is considered. In order to show the coherence of the beamformer direction across frequencies, the signal received at destination r from source q is assumed to be at an electrical angle $\theta_{qr} + \epsilon$, where ϵ is a small Gaussian random variable that varies across different frequencies. The sources have 6, 5 and 3 antennas and their corresponding destinations have 4, 7 and 4 antennas respectively placed as a uniform linear array as shown in Figure 3.2. Figure 3.3a shows the information rate achieved by each user against iteration index for the system shown in Figure 3.2, with $\theta_{11} = 5\pi/6, \theta_{12} = \pi/6, \theta_{13} = \pi/4, \theta_{21} = -\pi/6, \theta_{22} = 7\pi/6, \theta_{23} = -\pi/4, \theta_{31} = 2\pi/3, \theta_{32} = 5\pi/6$ and $\theta_{33} = \pi/3$. As seen in the figure, the broadband MIMO waterfilling algorithm has very fast convergence as expected from [99]. The information rates of the users reach their equilibrium values with 2-3 iterations.

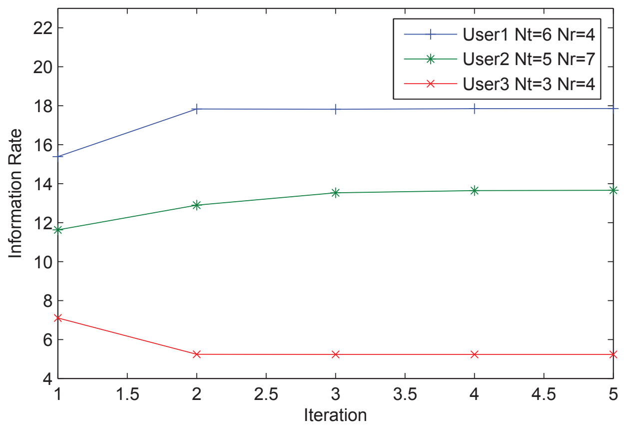
Figure 3.3b shows the plot of the beamforming patterns associated with the dominant eigenvalues of the optimal covariance matrices computed using the broadband MIMO iterative waterfilling algorithm for the above setup. The major lobes are in the direction of the appropriate receiver at all frequencies for both users. There are no side lobes as the system considered does not have multipath propagation. The fast convergence and appropriate beamformer directions have been observed for many channel realizations and different number of users and frequencies.

3.7 Effect of CSI estimation errors

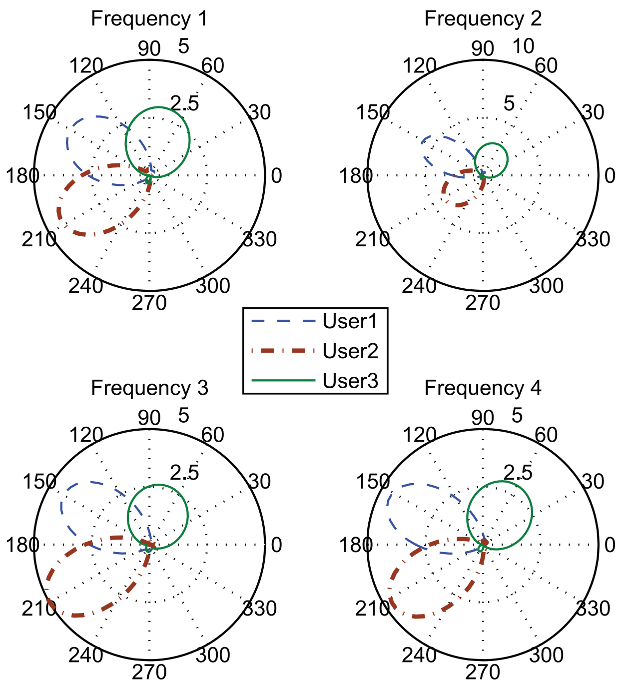
In this section, the effect of errors in the CSI on the Nash equilibrium of the MIMO iterative waterfilling algorithm is investigated. The MIMO iterative waterfilling algorithm [99] assumes that the direct channel matrices $\mathbf{H}_{qq}$ can be estimated accurately, which is not possible in practical systems. The estimate of the direct channel matrices in the presence of errors, $\widetilde{\mathbf{H}}_{qq}$, can be written as

$$\widetilde{\mathbf{H}}_{qq} = \mathbf{H}_{qq} + \mathbf{N}_q, \tag{3.7.1}$$

where $\mathbf{N}_q$ is a $n_{R_q} \times n_{T_q}$ matrix with zero mean complex Gaussian random elements of variance σ^2. The optimal covariance matrices $\{\widetilde{\mathbf{Q}}_q^*\}_{q\in\Omega}$ are computed based on the channel matrices $\widetilde{\mathbf{H}}_{qq}$ using the waterfilling algorithm

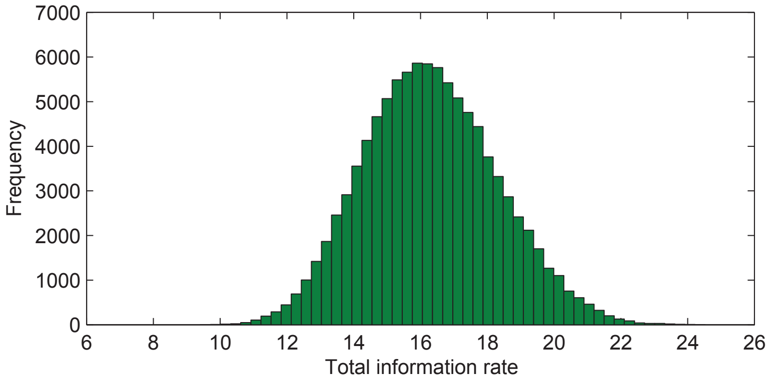


(a) Total information-rates of the two links v/s iteration index

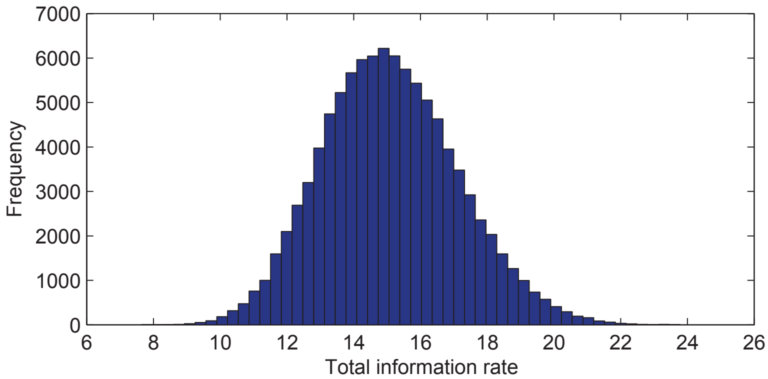


(b) Beamforming patterns of the three links at the Nash equilibrium of game $\mathscr{G}^{BM}$

Figure 3.3: System with two links with 4 antennas each and $\theta_{11} = 5\pi/6, \theta_{12} = \pi/6, \theta_{13} = \pi/4, \theta_{21} = -\pi/6, \theta_{22} = 7\pi/6, \theta_{23} = -\pi/4, \theta_{31} = 2\pi/3, \theta_{32} = 5\pi/6$ and $\theta_{33} = \pi/3$.



(a) Without CSI errors



(b) With CSI errors

Figure 3.4: Histograms of total information rates achieved using the MIMO iterative waterfilling algorithm (a) in the absence of CSI estimation errors and (b) in the presence of CSI estimation errors. It can be seen that the information rate achieved by the algorithm is reduced in the presence of CSI estimation errors.

and the total information rate achieved by all users with these covariance matrices is compared with the total information rates that would have been achieved in the absence of CSI estimation errors.

Consider a two user case, each with 4 receive and 4 transmit antennas.

The elements of the channel matrices $\mathbf{H}_{qr}$ are complex Gaussian random variables of unit variance and $\sigma = 0.5$. This setup is run for 100,000 different channel realizations. The histograms of the total information rates achieved in the absence and presence of CSI estimation errors are presented in Figures 3.4a and 3.4b respectively. It is clear that CSI estimation errors have a significant effect on the performance of the MIMO iterative waterfilling algorithm. The average loss in performance is about 7.45%. This indicates the necessity for the study of the robust distributed algorithms for optimal power allocation in the presence of CSI errors.

3.8 Summary

In this chapter, the conceptual and mathematical foundations from different areas needed for the techniques proposed in this thesis were summarized. This included results from contraction and fixed point theory, information theory (Gaussian channels and waterfilling) and the main game-theoretic approach to competitive rate-maximization in Gaussian interference channels. A brief investigation into the effect of channel state information errors on the performance of the MIMO iterative waterfilling algorithm was also presented, setting the scene for the robust solutions proposed in the subsequent chapters.

Appendix 3.A KKT Conditions

A convex optimization problem can be defined in the standard form as

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \le 0, \qquad i = 1, \ldots, m, \\ & h_i(\mathbf{x}) = 0, \qquad i = 1, \ldots, p, \end{aligned} \tag{3.A.1}$$

where the vector $\mathbf{x} \in \mathbb{R}^n$ is the *optimization variable* of the problem, the functions $f_0, \cdots, f_m$ are convex functions and the functions $h_1, \cdots, h_p$ are linear functions. The function f_0 is the *objective function* or *cost function*. The inequalities $f_i(\mathbf{x}) \le 0$ are called the *inequality constraints* and equalities $h_i(\mathbf{x}) = 0$ are called the *equality constraints*. If there are no constraints, then the problem is said to be an unconstrained problem. The *domain* of the optimization problem (3.A.1) is the set of points for which the objective and the constraints are defined and is denoted as

$$D = \bigcap_{i=0}^{m} \mathrm{dom} f_i \;\cap\; \bigcap_{i=0}^{p} \mathrm{dom} h_i. \tag{3.A.2}$$

A point $\mathbf{x} \in D$ is feasible, if it satisfies all the constraints $f_i(\mathbf{x}) \le 0 \;\; i = 1, \cdots, m$ and $h_i(\mathbf{x}) = 0 \;\; i = 1, \cdots, p$. Problem (3.A.1) is said to be *feasible* if at least one feasible point exists and is *infeasible* otherwise. The *optimal value* or the *solution* of the optimization problem is achieved at the optimal point $\mathbf{x}^\star$ if and only if

$$f_0(\mathbf{x}^\star) \leq f_0(\mathbf{x}) \quad \forall \mathbf{x} \in D. \tag{3.A.3}$$

The *Lagrangian* $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}$ for the original problem in (3.A.1) can be defined as [51]

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) + \sum_{i=1}^{p} \nu_i h_i(\mathbf{x}), \tag{3.A.4}$$

where λ_i and ν_i are the *Lagrange multipliers* associated with the ith inequality constraint $f_i(\mathbf{x}) \leq 0$ and equality constraint $h_i(\mathbf{x}) = 0$ respectively. The objective $f_0(\mathbf{x})$ in (3.A.1) is called the *primal objective* and the optimization variable $\mathbf{x}$ is termed the *primal variable.* Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ associated with the problem (3.A.4) are called the *dual variables*.

The Karush–Kuhn–Tucker conditions (also known as the Kuhn–Tucker or KKT conditions) are given by [51]:

1. Primal feasibility:

$$f_i(\mathbf{x}) \leq 0 \qquad i = 1, 2, \ldots, m, \tag{3.A.5}$$

$$h_i(\mathbf{x}) = 0 \qquad i = 1, 2, \ldots, p. \tag{3.A.6}$$

2. Dual feasibility:

$$\lambda_i \geq 0 \qquad i = 1, 2, \ldots, m. \tag{3.A.7}$$

3. Complementary slackness:

$$\lambda_i f_i(\mathbf{x}) = 0 \qquad i = 1, 2, \ldots, m. \tag{3.A.8}$$

4. Stationarity:

$$\nabla f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i \nabla f_i(\mathbf{x}) + \sum_{i=1}^{p} \nu_i \nabla h_i(\mathbf{x}) = 0. \tag{3.A.9}$$

The KKT conditions are necessary conditions for optimality in general but not sufficient conditions.

Chapter 4

ROBUST IWFA FOR SISO FREQUENCY-SELECTIVE SYSTEMS

In the previous chapter, the rate-maximization games for frequency-selective and MIMO Gaussian interference channels were introduced and the adverse effect of uncertainty in channel state information on the performance of these games was demonstrated. The results therein indicate that there is a necessity for the development of *robust* rate-maximization games that can perform well under channel state information uncertainty. In this chapter, an analytical framework for the robust rate-maximization game for a frequency-selective Gaussian interference channel is developed and characterized.

The chapter begins with a discussion of other work in the literature which has addressed the channel uncertainty issue in rate-maximization games. This is followed by the introduction of a distribution-free robust framework for rate-maximization games under bounded channel uncertainty. The closed-form equilibrium solution of this game is then derived, followed by analytical proofs for existence and uniqueness of the equilibrium. An asynchronous iterative algorithm to compute the equilibrium is then proposed. Finally, simulation results to demonstrate the behaviour of the algorithm are presented.

4.1 Related work

Uncertainty in game-theoretic and distributed-optimization problems in wireless communications has only recently been investigated. The issue of bounded uncertainty in specific distributed optimization problems in communication networks has been investigated in [101] wherein techniques to define the uncertainty set such that they can be solved distributively by robust optimization techniques are presented.

A robust optimization approach for the rate-maximization game with uncertainty in the noise-plus-interference estimate has been briefly investigated in [102], where the authors present a numerically computed algorithm unlike the closed form results presented in this chapter. Such a numerical solution prevents further mathematical analysis of the equilibrium and its behaviour under different uncertainty bounds. Also, this uncertainty model is different from the one adopted in this work, where the availability of CSI of the interfering channels with bounded uncertainty is assumed.

A similar problem of rate-maximization in the presence of uncertainty in the estimate of noise-plus-interference levels due to quantization in the feedback channel has been considered in [103]. This problem has been solved using a probabilistically constrained optimization approach and as in the work presented in this chapter, also results in the waterfilling solution moving closer to an FDMA solution, with corresponding improvement in sum-rate. However, the effect of quantization on the conditions for existence and uniqueness of the equilibrium and convergence of the algorithm have not been considered. The results presented in [103] are for a sequentially updated algorithm whereas the results of this chapter allow asynchronous (and thus sequential or simultaneous) updates to the algorithm. Also, the power allocations computed by such a probabilistic optimization formulation do not guarantee that the information rates expected will be achieved for all channel realizations, unlike the worst-case optimization formulation in this chapter. Furthermore, the relative error (and not just the absolute error due to quantization) in the interference estimate as defined in [103] is assumed to be bounded and drawn from a uniform distribution, which is inaccurate. In addition, this bound on the relative error can only be computed if the noise variance at the receivers is assumed to be known (which is not the case). The bounds computed in such a fashion are very loose and will degrade system performance. The other assumption that this relative error bound is in the range $[0, 1)$ means that the absolute quantization error has to be less than the noise variance at the receivers, which restricts the applicability of the approach. The problem formulation in this chapter has no such limitation on the uncertainty bound based on the noise variances in the system.

Dynamic robust games for MIMO systems, where a learning framework is used to develop suitable power allocations in repeated games with channel uncertainty and delayed imperfect payoffs (information rate), have been recently proposed in [104]. A robust rate-maximization game for a cognitive radio scenario with uncertainty in the channel to the primary user has been presented in [105]. This leads to a noncooperative game formulation without any uncertainties in the payoff functions of the game (unlike the game formulation in this chapter) with robust interference limits acting as a constraint on the admissible set of strategies. This game is then solved using numerical techniques as there is no closed form solution.

4.2 System model

Consider a system similar to the one in [93], which is a SISO frequency-selective Gaussian interference channel with N frequencies, composed of Q SISO links. $\Omega \triangleq \{1, \ldots, Q\}$ is the set of the Q players (i.e. SISO links). The quantity $H_{rq}(k)$ denotes the complex frequency response of the k-th frequency bin of the channel between source r and destination q. The variance of the zero-mean circularly symmetric complex Gaussian noise at receiver q in the frequency bin k is denoted by $\bar{\sigma}_q^2(k)$. The channel is assumed to be quasi-stationary for the duration of the transmission. Let $\sigma_q^2(k) \triangleq \bar{\sigma}_q^2(k)/|H_{qq}(k)|^2$ and the total transmit power of user q be P_q. Let the vector $\mathbf{s}_q \triangleq [s_q(1)\ s_q(2) \ldots s_q(N)]$ be the N symbols transmitted by user q on the N frequency bins and $p_q(k) \triangleq \mathrm{E}\{|s_q(k)|^2\}$ be the power allocated to the k-th frequency bin by user q and $\mathbf{p}_q \triangleq [p_q(1)\ p_q(2) \dots p_q(N)]$ be the power allocation vector. The power allocation of each user q has two constraints:

- Maximum total transmit power for each user

$$\mathrm{E}\left\{\|\mathbf{s}_q\|_2^2\right\} = \sum_{k=1}^{N} p_q(k) \leq P_q, \tag{4.2.1}$$

  for $q = 1, \dots, Q$, where P_q is power in units of energy per transmitted symbol.

- Spectral mask constraints

$$\mathrm{E}\left\{|s_q(k)|^2\right\} = p_q(k) \leq p_q^{max}(k), \tag{4.2.2}$$

  for $k = 1, \dots, N$ and $q = 1, \dots, Q$, where $p_q^{max}(k)$ is the maximum power that is allowed to be allocated by user q for the frequency bin k.

Each receiver estimates the channel between itself and all the transmitters, which is private information. The power allocation vectors are public information, i.e. known to all users. Each receiver computes the optimal power allocation across the frequency bins for its own link and transmits it back to the corresponding transmitter in a low bit-rate error-free feedback channel. Note that this leads to sharing of more information compared to other work in the literature such as [93]. The channel state information estimated by each receiver is assumed to have a bounded uncertainty of unknown distribution. An ellipsoid is often used to approximate complicated convex uncertainty sets [51]. The ellipsoidal approximation has the advantage of parametrically modelling complicated data sets and thus provides a convenient input parameter to algorithms. Further, in certain cases there are statistical reasons leading to ellipsoidal uncertainty sets and also results in optimization problems with convenient analytical structures [60, 106].

At each frequency, the uncertainty in the channel state information of each user is deterministically modelled under an ellipsoidal approximation[1]

$$\mathcal{F}_q = \left\{ F_{rq}(k) + \Delta F_{rq,k} \ : \ \sum_{r \neq q} |\Delta F_{rq,k}|^2 \leq \epsilon_q^2 \quad \forall k = 1, \ldots, N \right\}, \tag{4.2.3}$$

where $\epsilon_q \geq 0 \ \forall \ q \in \Omega$ is the uncertainty bound and

$$F_{rq}(k) \triangleq \frac{|H_{rq}(k)|^2}{|H_{qq}(k)|^2}, \tag{4.2.4}$$

with $F_{rq}(k)$ being the nominal value. It is possible to consider uncertainty in $F_{rq}(k)$ instead of $H_{rq}(k)$ because bounded uncertainties in $F_{rq}(k)$ and $H_{rq}(k)$ are equivalent, but with different bounds.[2]

[1] More specifically, the uncertainty set in (4.2.3) is a spherical approximation.

[2] The model considered here has some redundancy in the uncertainty for the case when $F_{rq}(k) = 0$ which leads to including $F_{rq}(k) + \Delta F_{rq,k} < 0$ in the model which can never happen in practice. However, this does not affect the solution in this method due to the nature of the max-min problem formulation in (4.3.3) which leads to selection of positive values of $\Delta F_{rq,k}$.

The nominal information rate of user q can be written as [64]

$$R_q = \sum_{k=1}^{N} \log\left(1 + \frac{p_q(k)}{\sigma_q^2(k) + \sum_{r\neq q} F_{rq}(k) p_r(k)}\right), \tag{4.2.5}$$

where $\sigma_q^2(k) \triangleq \bar{\sigma}_q^2(k)/|H_{qq}(k)|^2$.

A robust rate-maximization game is formulated and analyzed based on this system model in the subsequent sections of this chapter and in Chapter 5.

4.3 Robust rate-maximization game formulation

According to the robust game model presented in Section 2.5 on page 47, each player formulates a best response as the solution of a robust (worst-case) optimization problem for the uncertainty in the payoff function (information rate), given the other players' strategies. If all the players know that everyone else is using the robust optimization approach to the payoff uncertainty, they would then be able to mutually predict each other's behaviour. The robust game $\mathscr{G}_{\text{rob}}^{S}$ where each player q formulates a worst-case robust optimization problem can be written as, $\forall\, q \in \Omega$,

$$\begin{aligned} \max_{\mathbf{p}_q} \; & \min_{\tilde{F}_{rq} \in \mathcal{F}_q} \sum_{k=1}^{N} \log\left(1 + \frac{p_q(k)}{\sigma_q^2(k) + \sum\limits_{r\neq q} \tilde{F}_{rq}(k) p_r(k)}\right) \\ \text{s.t.} \; & \mathbf{p}_q \in \mathcal{P}_q, \end{aligned} \tag{4.3.1}$$

where $\mathcal{F}_q$ is the uncertainty set which is modelled under ellipsoid approximation as shown in (4.2.3), $\Omega \triangleq \{1, \ldots, Q\}$ is the set of the Q players (i.e.

the SISO links) and $\mathcal{P}_q$ is the set of admissible strategies of user q, which is defined as

$$\mathcal{P}_q \triangleq \left\{ \mathbf{p}_q \in \mathbb{R}^N : \sum_{k=1}^{N} p_q(k) = P_q,\ 0 \le p_q(k) \le p_q^{max}(k),\ k = 1, \dots, N \right\}. \tag{4.3.2}$$

This optimization problem using uncertainty sets can be equivalently written in a form represented by protection functions [101] as, $\quad \forall q \in \Omega$,

$$\begin{aligned} &\max_{\mathbf{p}_q} \min_{\Delta F_{rq,k}} \sum_{k=1}^{N} \log \left(1 + \frac{p_q(k)}{\sigma_q^2(k) + \sum\limits_{r \neq q} (F_{rq}(k) + \Delta F_{rq,k}) p_r(k)} \right) \\ &\qquad \text{s.t.} \ \sum_{r \neq q} |\Delta F_{rq,k}|^2 \le \epsilon_q^2, \quad k = 1, ...N \\ &\qquad \phantom{\text{s.t.}} \ \mathbf{p}_q \in \mathcal{P}_q. \end{aligned} \tag{4.3.3}$$

Using the Cauchy-Schwarz inequality [107], $\forall k = 1, \dots, N$,

$$\sum_{r \neq q} \Delta F_{rq,k} p_r(k) \ \le \ \left[\sum_{r \neq q} |\Delta F_{rq,k}|^2 \sum_{r \neq q} |p_r(k)|^2 \right]^{\frac{1}{2}} \tag{4.3.4}$$

$$\le \ \epsilon_q \sqrt{\textstyle\sum_{r \neq q} p_r^2(k)} \tag{4.3.5}$$

Using (4.3.5), the robust game can be formulated as, $\quad \forall q \in \Omega$,

$$(\mathscr{G}_{\text{rob}}^S) : \quad \begin{aligned} &\max_{\mathbf{p}_q} \sum_{k=1}^{N} \log \left(1 + \frac{p_q(k)}{\sigma_q^2(k) + \sum\limits_{r \neq q} F_{rq}(k) p_r(k) + \epsilon_q \sqrt{\sum\limits_{r \neq q} p_r^2(k)}} \right) \\ &\text{s.t.} \quad \mathbf{p}_q \in \mathcal{P}_q. \end{aligned} \tag{4.3.6}$$

Having defined the problem for robust rate-maximization under bounded channel uncertainty, the solution to the optimization problem in (4.3.6) for a single-user is presented in the following section.

4.4 Robust waterfilling solution

The closed-form solution to the robust optimization problem in (4.3.6) for any particular user q is given by the following theorem:

THEOREM 4.1. *Given the set of power allocations of other users* $\mathbf{p}_{-q} \triangleq \{\mathbf{p}_1, \ldots, \mathbf{p}_{q-1}, \mathbf{p}_{q+1}, \ldots, \mathbf{p}_Q\}$*, the solution to the robust optimization problem of user* q*,*

$$\begin{aligned}&\max_{\mathbf{p}_q} \sum_{k=1}^{N} \log\left(1+\frac{p_q(k)}{\sigma_q^2(k)+\sum\limits_{r\neq q} F_{rq}(k)p_r(k)+\epsilon_q\sqrt{\sum\limits_{r\neq q} p_r^2(k)}}\right)\\ &\text{s.t.}\quad \mathbf{p}_q \in \mathcal{P}_q.\end{aligned} \tag{4.4.1}$$

is given by the waterfilling solution

$$\mathbf{p}_q^{\star} = \mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}}(\mathbf{p}_{-q}), \tag{4.4.2}$$

where the waterfilling operator $\mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}}(\cdot)$ *is defined for* $k = 1, \ldots, N$*, as*

$$\left[\mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}}(\mathbf{p}_{-q})\right]_k \triangleq \left[\mu_q - \sigma_q^2(k) - \sum_{r\neq q} F_{rq}(k)p_r(k) - \epsilon_q\sqrt{\sum_{r\neq q} p_r^2(k)}\right]_0^{p_q^{max}(k)} \tag{4.4.3}$$

where μ_q *is chosen to satisfy the power constraint* $\sum_{k=1}^{N} p_q^{\star}(k) = 1$.

Proof. See Appendix 4.A. □

The robust waterfilling operation for each user is a distributed worst-case optimization under bounded channel uncertainty. Compared with the original waterfilling operation in [93] under perfect CSI (i.e. $\epsilon_q \equiv 0$), it can be seen that an additional term has appeared in (4.4.3) for $\epsilon_q > 0$.

This additional term can be interpreted as a penalty for allocating power to frequencies having a large product of uncertainty bound and norm of the powers of the other players currently transmitting in those frequencies. This is because the users assume the worst-case interference from other users and are thus conservative about allocating power to such channels where there is a strong presence of other users.

4.4.1 Robust waterfilling as a projection operation

Let $\Phi_q(k)$ represent the denominator terms in (4.4.1), which is the worst-case noise-plus-interference

$$\Phi_q(k) \triangleq \sigma_q^2(k) + \sum_{r \neq q} F_{rq}(k) p_r(k) + \epsilon_q \sqrt{\sum_{r \neq q} p_r^2(k)}. \tag{4.4.4}$$

It has been shown in [92] that the waterfilling operation can be interpreted as the Euclidean projection of a vector onto a simplex. Using this framework, the robust waterfilling operator in (4.4.3) can be expressed as the Euclidean projection of the vector $\mathbf{\Phi}_q \triangleq [\Phi_q(1), \ldots, \Phi_q(N)]^T$ onto the simplex $\mathcal{P}_q$ defined in (4.3.2):

$$\mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}}(\mathbf{p}_{-q}) = \left[-\boldsymbol{\Phi}_q\right]_{\mathcal{P}_q}, \tag{4.4.5}$$

which can be conveniently written as

$$\mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}}(\mathbf{p}_{-q}) = \Big[-\boldsymbol{\sigma}_q - \sum_{r\neq q}\mathbf{F}_{rq}\mathbf{p}_r - \epsilon_q\mathbf{f}_q\Big]_{\mathcal{P}_q}, \tag{4.4.6}$$

where

$$\boldsymbol{\sigma}_q \triangleq \Big[\sigma_q^2(1),\ldots,\sigma_q^2(N)\Big]^T, \tag{4.4.7}$$

$$\mathbf{F}_{rq} \triangleq \mathrm{Diag}\Big(F_{rq}(1),\ldots,F_{rq}(N)\Big), \tag{4.4.8}$$

$$\mathbf{f}_q \triangleq \Big[\sqrt{\textstyle\sum_{r\neq q} p_r^2(1)}\ ,\ldots,\sqrt{\textstyle\sum_{r\neq q} p_r^2(N)}\Big]^T. \tag{4.4.9}$$

Let $\mathcal{N} = \{1,\ldots,N\}$ be the set of frequency bins. Let $\mathcal{D}_q^\circ$ denote the set of frequency bins that user q would never use as the best response to any set of strategies adopted by the other users,

$$\mathcal{D}_q^\circ \triangleq \Big\{k \in \{1,\ldots,N\} : \left[\mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}}(\mathbf{p}_{-q})\right]_k = 0 \quad \forall \mathbf{p}_{-q} \in \mathcal{P}_{-q}\Big\} \tag{4.4.10}$$

where $\mathcal{P}_{-q} \triangleq \mathcal{P}_1 \times \cdots \times \mathcal{P}_{q-1} \times \mathcal{P}_{q+1} \times \cdots \times \mathcal{P}_Q$. The non-negative matrices $\mathbf{E}$ and $\mathbf{S}^{max} \in \mathbb{R}_+^{Q\times Q}$ are defined as

$$[\mathbf{E}]_{qr} \triangleq \begin{cases} \epsilon_q, & \text{if } r \neq q, \\ 0, & \text{otherwise,} \end{cases} \tag{4.4.11}$$

and

$$[\mathbf{S}^{max}]_{qr} \triangleq \begin{cases} \max\limits_{k \in \mathcal{D}_q \cap \mathcal{D}_r} F_{rq}(k), & \text{if } r \neq q, \\ 0, & \text{otherwise}, \end{cases} \tag{4.4.12}$$

where $\mathcal{D}_q$ is any subset of $\{1, \dots, N\}$ such that $\mathcal{N} - \mathcal{D}_q^{\circ} \subseteq \mathcal{D}_q \subseteq \{1, \dots, N\}$.

Contraction property of the waterfilling projection

The contraction property of the waterfilling mapping is given by the following lemma:

Lemma 4.1. *Given* $\mathbf{w} \triangleq [w_1, \dots, w_Q]^T > \mathbf{0}$, *the mapping* $\mathrm{RWF}^{\mathrm{S}}(\cdot)$ *defined in* (4.B.1) *satisfies*

$$\| \mathrm{RWF}^{\mathrm{S}}\,(\mathbf{p}^{(1)}) - \mathrm{RWF}^{\mathrm{S}}\,(\mathbf{p}^{(2)}) \|_{2,\mathrm{block}}^{\mathbf{w}} \leq ||\mathbf{S}^{max} + \mathbf{E}||_{\infty,\mathrm{mat}}^{\mathbf{w}} \times \|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\|_{2,\mathrm{block}}^{\mathbf{w}}, \tag{4.4.13}$$

$\forall \mathbf{p}^{(1)}, \mathbf{p}^{(2)} \in \mathcal{P}$, $\mathbf{E}$ *and* $\mathbf{S}$ *as defined in* (4.4.11) *and* (4.4.12) *respectively. Furthermore, if*

$$||\mathbf{S}^{max} + \mathbf{E}||_{\infty,\mathrm{mat}}^{\mathbf{w}} < 1, \tag{4.4.14}$$

for some $\mathbf{w} > \mathbf{0}$, *then the mapping* $\mathrm{RWF}^{\mathrm{S}}(\cdot)$ *is a block contraction with modulus* $\alpha = ||\mathbf{S}^{max} + \mathbf{E}||_{\infty,\mathrm{mat}}^{\mathbf{w}}$.

Proof. See Appendix 4.B. □

Having derived and characterized the robust waterfilling solution in the presence of channel uncertainty, the issue of whether a stable equilibrium for the system exists is to be considered. If so, its properties and computation are to be considered. This is undertaken in the subsequent sections.

4.5 Robust-optimization equilibrium

The solution to the game $\mathscr{G}^S_{\text{rob}}$ is the robust-optimization equilibrium. At any robust-optimization equilibrium of this game, the optimum action profile of the players $\{\mathbf{p}^\star_q\}_{q\in\Omega}$ must satisfy the following set of simultaneous waterfilling equations: $\forall q \in \Omega$,

$$\mathbf{p}^\star_q = \mathrm{RWF}^{\mathrm{S}}_{\mathrm{q}}(\mathbf{p}^\star_1, \ldots, \mathbf{p}^\star_{q-1}, \mathbf{p}^\star_{q+1}, \ldots, \mathbf{p}^\star_Q) = \mathrm{RWF}^{\mathrm{S}}_{\mathrm{q}}(\mathbf{p}^\star_{-q}). \tag{4.5.1}$$

It can easily be verified that the robust-optimization equilibrium reduces to the Nash equilibrium of the system [93] when there is no uncertainty in the system. In Chapter 5, the global efficiency of the robust-optimization equilibrium for the two-user case is analyzed and it is shown that the robust-optimization equilibrium has a higher efficiency than the Nash equilibrium due to a penalty for interference which encourages better partitioning of the frequency space among the users.

4.5.1 Analysis of the equilibrium of game $\mathscr{G}^S_{\text{rob}}$

The contraction property of the waterfilling mapping is useful in the analysis of the equilibrium of game $\mathscr{G}^S_{\text{rob}}$. A sufficient condition for existence and uniqueness of the robust-optimization equilibrium of game $\mathscr{G}^S_{\text{rob}}$ is given by the following theorem:

Theorem 4.2. *Game* $\mathscr{G}_{\text{rob}}^{S}$ *has at least one equilibrium for any set of channel values and transmit powers of the users. Furthermore, the equilibrium is unique if*

$$\rho(\mathbf{S}^{max}) < 1 - \rho(\mathbf{E}), \tag{4.5.2}$$

where $\mathbf{E}$ *and* $\mathbf{S}$ *are as defined in* (4.4.11) *and* (4.4.12) *respectively.*

Proof. See Appendix 4.C. □

In the absence of uncertainty, i.e. when $\epsilon_q = 0 \ \forall q \in \Omega$, this condition reduces to condition (C1) in [93] as expected. Since $\rho(\mathbf{E}) \geq 0$, the condition on $\mathbf{S}^{max}$ becomes more stringent as the uncertainty bound increases, i.e. the set of channel coefficients for which the existence of a unique equilibrium is guaranteed shrinks as the uncertainty bound increases.

4.6 Iterative algorithm for robust waterfilling

In this section, an iterative waterfilling algorithm, based on the framework presented in Section 3.1.2, for computing the robust-optimization equilibrium is presented.

Let the discrete set $\mathbb{T} \subseteq \mathbb{N}_+ = \{1, 2, \dots\}$ be the set of times at which one or more users update their strategies and T be the number of iterations for which the algorithm is run. Let $\mathbf{p}_q^{(n)}$ denote the vector power allocation of user q at the discrete time n, and let $\mathbb{T}_q \subseteq \mathbb{T}$ denote the set of time instants n when the power vector $\mathbf{p}_q^{(n)}$ of user q is updated. Let $\tau_r^q(n)$ denote the time

Algorithm 4.1 – Robust SISO Iterative Waterfilling Algorithm

Input:

Ω: Set of users in the system

$\mathcal{P}_q$: Set of admissible strategies of user q

$\mathbb{T}_q$: Set of time instants n when the power vector $\mathbf{p}_q^{(n)}$ of user q is updated

T: Number of iterations for which the algorithm is run

$\tau_r^q(n)$: Time of the most recent power allocation of user r available to user q at time n

$\mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}}(\cdot)$: Robust waterfilling operation in (4.4.3)

Initialization: $n = 0$ and $\mathbf{p}_q^{(0)} \leftarrow$ any $\mathbf{p} \in \mathcal{P}_q,\ \forall q \in \Omega$

for $n = 0$ to T **do**

$$\mathbf{p}_q^{(n+1)} = \begin{cases} \mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}}\left(\mathbf{p}_{-q}^{(\tau^q(n))}\right), & \text{if } n \in \mathbb{T}_q, \\ \mathbf{p}_q^{(n)}, & \text{otherwise,} \end{cases} \quad \forall q \in \Omega.$$

end for

of the most recent power allocation information of user r which is available to user q at time n (Note that $0 \le \tau_r^q(n) \le n$). Hence, if the strategy of user q is updated at time n, then

$$\mathbf{p}_{-q}^{\left(\tau^q(n)\right)} \triangleq \left(\mathbf{p}_1^{\left(\tau_1^q(n)\right)}, \dots, \mathbf{p}_{q-1}^{\left(\tau_{q-1}^q(n)\right)}, \mathbf{p}_{q+1}^{\left(\tau_{q+1}^q(n)\right)}, \dots, \mathbf{p}_Q^{\left(\tau_Q^q(n)\right)}\right). \tag{4.6.1}$$

The robust asynchronous iterative waterfilling algorithm for computing the equilibrium of game $\mathscr{G}_{\mathrm{rob}}^S$ in a distributed fashion is described in Algorithm 4.1. The convergence of Algorithm 4.1 is guaranteed under the following sufficiency condition:

THEOREM 4.3. *The asynchronous iterative waterfilling algorithm described in Algorithm 4.1 converges to the unique robust-optimization equilibrium of* *game* $\mathscr{G}_{\mathrm{rob}}^{S}$ *as the number of iterations for which the algorithm is run,* $T \rightarrow \infty$ *for any set of feasible initial conditions if condition* (4.5.2) *is satisfied.*

Proof. See Appendix 4.D. □

The global convergence of the distributed robust iterative waterfilling algorithm to the unique robust-optimization equilibrium is guaranteed by Theorem 4.3 using condition (4.5.2) despite game $\mathscr{G}_{\mathrm{rob}}^{S}$ and the waterfilling operation $\mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}}(\cdot)$ being nonlinear. Also, from Lemma 4.1, the modulus of the waterfilling contraction increases as uncertainty increases. This indicates that the convergence of the iterative waterfilling algorithm becomes slower as the uncertainty increases, as seen in the simulation results in Section 4.7. Also, the set of channel coefficients for which convergence of the algorithm is guaranteed reduces as the uncertainty bound increases.

Corollary 4.3.1. *When the uncertainties of all the* Q *users are equal (say* ϵ*), the robust-optimization equilibrium of game* $\mathscr{G}_{\mathrm{rob}}^{S}$ *is unique and Algorithm 4.1 converges to the unique robust-optimization equilibrium of game* $\mathscr{G}_{\mathrm{rob}}^{S}$ *as* $T \rightarrow \infty$ *for any set of feasible initial conditions if*

$$\rho(\mathbf{S}^{max}) < 1 - \epsilon(Q-1) \tag{4.6.2}$$

Proof. When the uncertainties of all Q users is ϵ, $\rho(\mathbf{E}) = \epsilon(Q-1)$. □

The above corollary explicitly shows how the uncertainty bound and the number of users in the system affect the existence of the equilibrium and the convergence to the equilibrium using an iterative waterfilling algorithm. For a fixed uncertainty bound, as the number of users in the system increases, there is a larger amount of uncertain information in the system. Hence, the probability that a given system for a fixed uncertainty bound will converge will decrease as the number of users in the system increases. Also, if $\epsilon(Q-1) \geq 1$, a unique equilibrium for non-zero uncertainty bounds cannot be guaranteed regardless of the channel coefficients. This will help determine the number of users that could be allowed to operate in the system based on the uncertainty bounds.

The modified waterfilling operation in (4.4.3) can also be used as a pricing mechanism to achieve improved sum-rate performance in a system with no uncertainty where ϵ is a design parameter, with all the analytical results presented here still being valid.

4.7 Simulation results

In this section, the behaviour of the equilibrium under varying uncertainty bounds is investigated through numerical simulations. The simulations are computed for a system with Q users and N frequencies averaged over 5000 channel realizations. The channel gains are $H_{rq}(k) \sim N_C(0,1)$ for $r \neq q$ and $H_{qq}(k) \sim N_C(0,2.25)$. The channel uncertainty model used is nominal value $F_{rq}(k) = F_{rq}^{\text{true}}(k)(1+e_{rq}(k))$ with $e_{rq}(k) \sim U(-\frac{\delta}{2}, \frac{\delta}{2})$, $\delta < 1$. The specific parameter values used for the simulations are provided with each figure. In these simulations, the actual convergence of the algorithm for every channel realization is tested, i.e., the simulations are not limited only

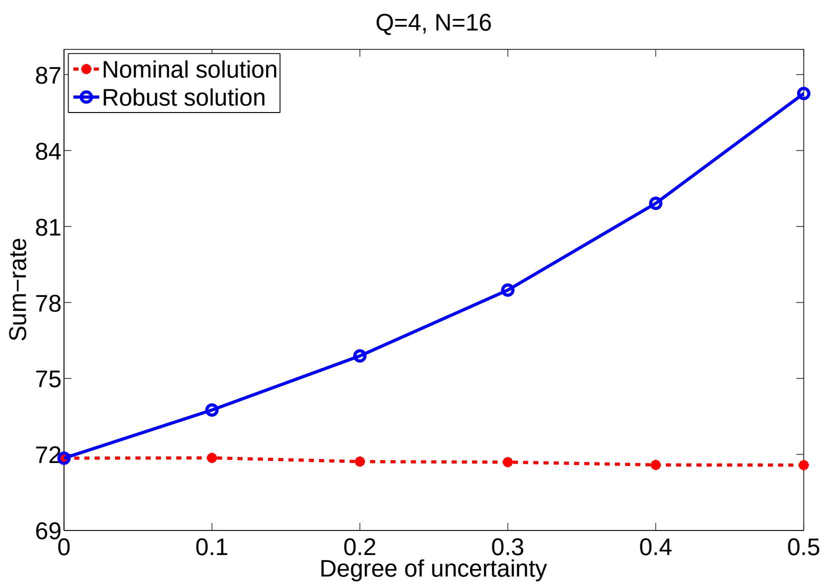


Figure 4.1: Sum-rate of the system vs. uncertainty δ.

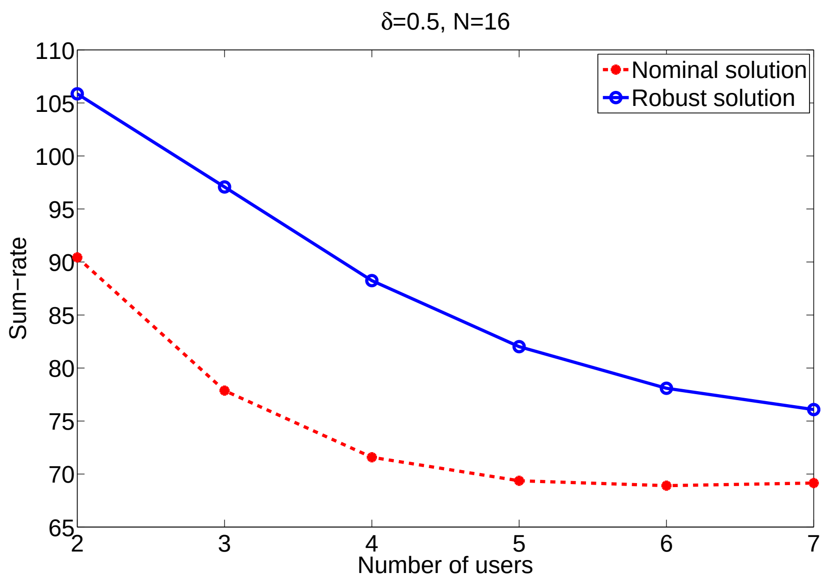


Figure 4.2: Sum-rate of the system vs. number of users, Q.

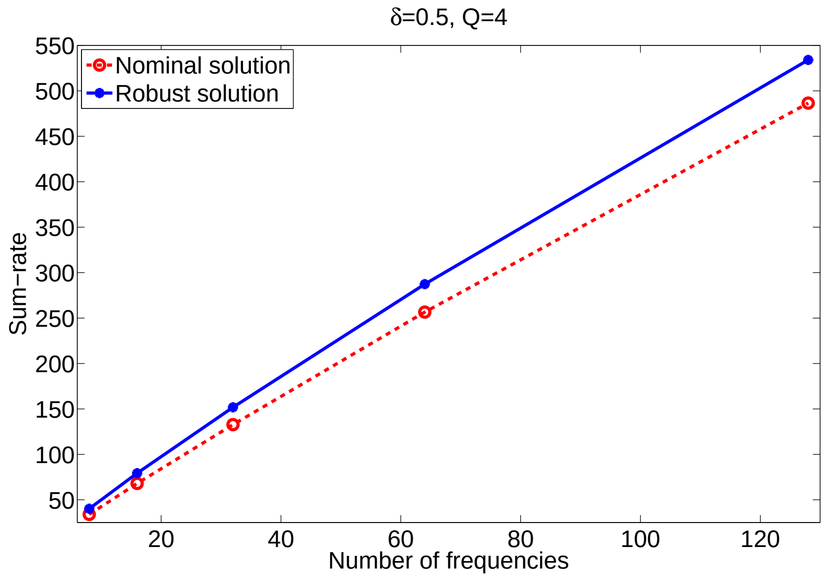


Figure 4.3: Sum-rate of the system vs. number of frequencies, N.

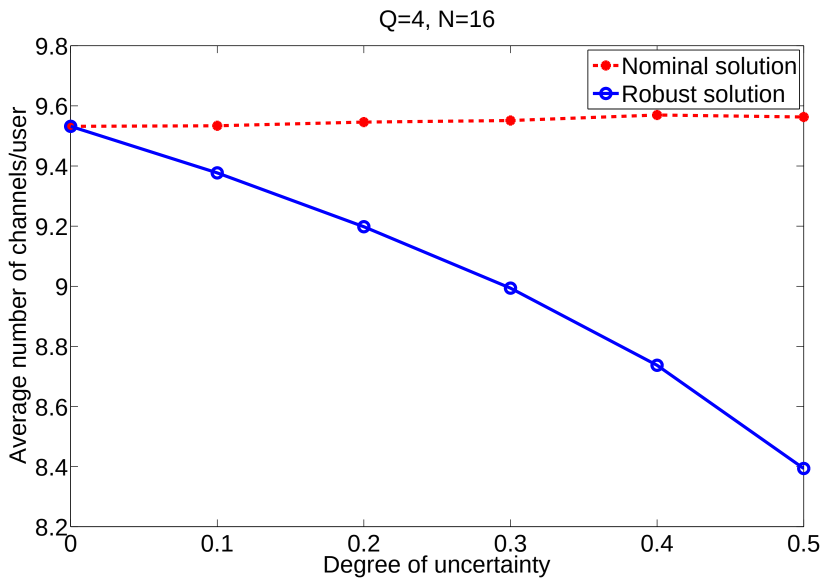


Figure 4.4: Average number of channels occupied per user vs. uncertainty, δ.

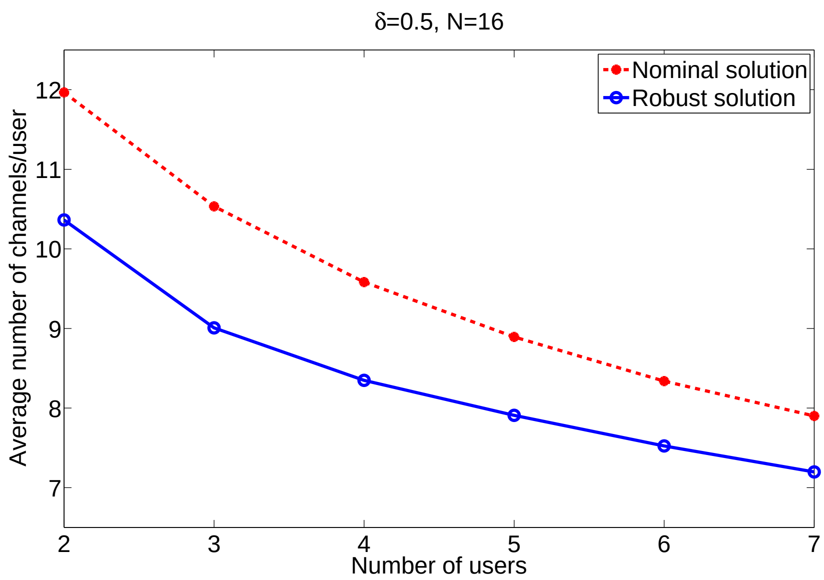


Figure 4.5: Average number of channels occupied per user vs. number of users, Q.

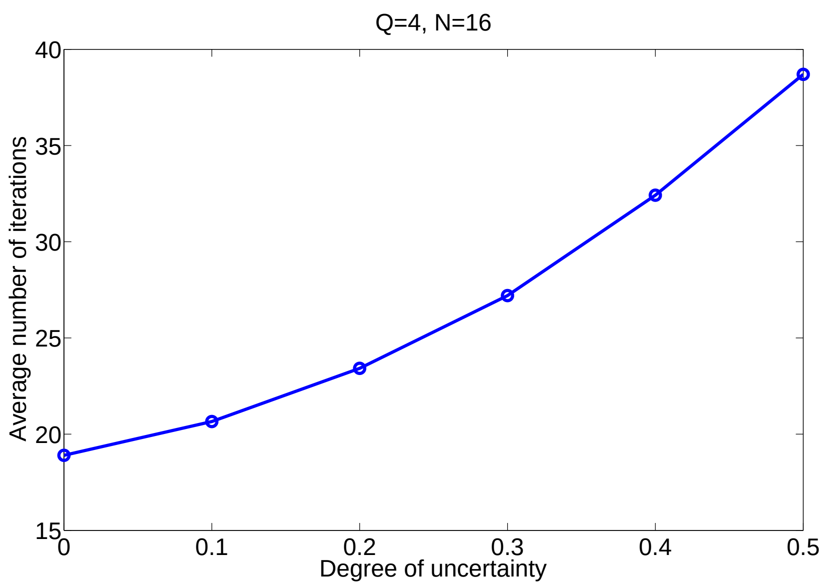


Figure 4.6: Average number of iterations vs. uncertainty, δ.

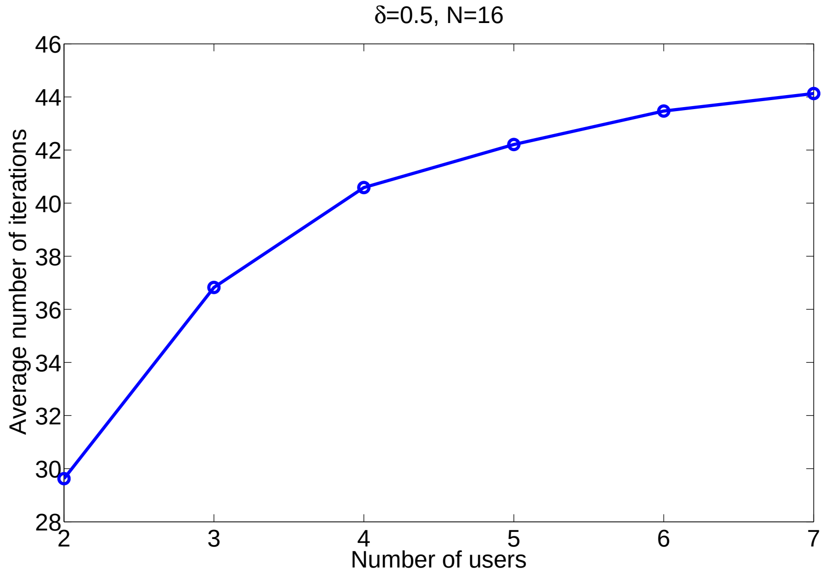


Figure 4.7: Average number of iterations vs. number of users, Q.

to channels satisfying the convergence condition (4.6.2). (Refer to Section 5.3 for simulations which are limited to channels satisfying (4.6.2). It can be seen that there is little difference in the average behaviour of the equilibrium.)

Note that the zero uncertainty solution corresponds to the Nash equilibrium and the nominal solution is the solution resulting from using the erroneous channel values in the traditional rate-maximization game $\mathscr{G}^S$ without accounting for its uncertainty. The effect of uncertainty, number of users and number of frequencies on the average sum-rate of the system, the average number of frequencies occupied by each user and the average number of iterations for convergence are examined. In these figures, the Nash equilibrium point is when the uncertainty is zero.

In Figure 4.1, it can be observed that the sum-rate at the Nash equilibrium under perfect CSI is less than the sum-rate at the robust-optimization equilibrium under imperfect CSI and that the gap in performance increases to about 20% as the uncertainty δ increases to 0.5. Under imperfect CSI, the power allocation using the robust-optimization equilibrium in (4.4.3) and (4.5.1) has higher sum-rate as uncertainty increases.[3] This is because the users are more cautious about using frequencies with significant interference, thus reducing the total amount of interference in the system.

In Figure 4.2, it can be observed that the sum-rate of the system under the robust solution drops from about 105 nats/transmission to about 80 nats/transmission when the number of users rises from 2 to 7. This is because having a greater number of users results in higher interference for all users and this effect is strong enough to counter user diversity which would have resulted in higher sum-rates if the users were on an FDMA scheme. In Figure 4.3, it can be observed that the sum-rate of the system improves with increase in number of frequencies and also that the robust solution continues to perform better than the nominal solution even when the number of frequencies increases.

In Figure 4.4, it can be seen that the robust solution results in a lower average number of channels per user as the uncertainty, δ increases. Also, the total number of channels each user occupies at the robust-optimization equilibrium is less than at the nominal solution, regardless of the number of users, as can be seen in Figure 4.5. This implies that the users are using a

[3](4.4.3) and (4.5.1) are in terms of absolute uncertainty ϵ while the simulations use relative uncertainty δ. They are equivalent to one another.

smaller number of frequencies, which demonstrates the better partitioning of the frequency space among the users to reduce interference. Hence, this leads to the higher sum-rates as observed in Figure 4.1.

In Figures 4.6 and 4.7, it can be observed that the average number of iterations for convergence increases as the uncertainty δ and the number of users Q increase respectively. This is as expected from Lemma 4.1 and Corollary 4.3.1, as the modulus of the block-contraction in (4.4.13) increases as the uncertainty increases. This indicates that the step size of each iteration reduces as uncertainty increases, leading to slower convergence. Thus, the trade-off for robust solutions with higher sum-rates is in a higher number of iterations before convergence.

4.8 Summary

In this chapter, a robust framework for rate-maximization games under bounded channel uncertainty was developed. After defining the robust rate-maximization game, closed-form expressions for the equilibrium solution were derived. An asynchronous iterative waterfilling algorithm for computing the equilibrium was proposed. Sufficient conditions for the existence and uniqueness of the equilibrium and convergence of the iterative algorithm to the equilibrium were presented. Finally, simulation results demonstrating the effect of uncertainty on the performance of the game were presented. The interesting effect of improvement in sum-rate with higher channel uncertainty bounds was observed in the simulation results and is the focus of analysis in the next chapter.

Appendix 4.A Proof of Theorem 4.1

The optimization problem in (4.4.1) is convex and admits a unique solution since the cost function is strictly concave for $\mathbf{p}_q > \mathbf{0}$. The Lagrangian $\mathcal{L}$ for this problem can be written as,

$$\begin{aligned}\mathcal{L} = \sum_{k=1}^{N} \log\left(1 + \frac{p_q(k)}{\Phi_q(k)}\right) \; - \; \frac{1}{\mu_q}\Big(\sum_{k=1}^{N} p_q(k) - 1\Big) \\ - \sum_{k=1}^{N} v_k p_q(k) + \sum_{k=1}^{N} \lambda_k \left(p_q(k) - p_q^{max}(k)\right),\end{aligned} \tag{4.A.1}$$

where $\Phi_q(k)$ is defined in (4.4.4). Note that $\Phi_q(k)$ is independent of the optimization variable $p_q(k)$.

The KKT optimality conditions [Refer Appendix 3.A] for problem (4.4.1) are

$$\left(1 + \frac{p_q(k)}{\Phi_q(k)}\right)^{-1} \frac{1}{\Phi_q(k)} - \frac{1}{\mu_q} - v_k + \lambda_k = 0, \tag{4.A.2}$$

$$\lambda_k \geq 0; \quad v_k \geq 0, \tag{4.A.3}$$

$$p_q(k) \geq 0; \quad p_q(k) \leq p_q^{max}(k), \tag{4.A.4}$$

$$\lambda_k \left(p_q(k) - p_q^{max}(k)\right) = 0; \quad v_k p_q(k) = 0, \tag{4.A.5}$$

$$\sum_{k=1}^{N} p_q(k) = 1. \tag{4.A.6}$$

for $k = 1, \ldots, N$.

To avoid the problem becoming infeasible or having the trivial solution $p_q(k) = p_q^{max}(k)\ \forall k$, it is assumed that $\sum_{k=1}^{N} p_q^{max} > 1$. From constraint (4.A.6), $p_q(k) > 0$ for at least one k. Since $\Phi_q(k) > 0$, it implies that $-1/\mu_q - v_k + \lambda_k < 0$. Further, at any instance, at most one among λ_k and v_k can be non-zero for a feasible solution, based on the complementary slackness conditions in (4.A.5) which implies that $\mu_q > 0$.

When $0 < p_q(k) < p_q^{max}(k)$, $\lambda_k = 0$ and $v_k = 0$, from the complementary slackness conditions in (4.A.5). Thus, from (4.A.2)

$$\left(1 + \frac{p_q(k)}{\Phi_q(k)}\right)^{-1} \frac{1}{\Phi_q(k)} - \frac{1}{\mu_q} = 0, \tag{4.A.7}$$

which gives

$$p_q(k) = \mu_q - \Phi_q(k), \tag{4.A.8}$$

where μ_q is chosen such that constraint (4.A.6) is satisfied. Substituting for $\Phi_q(k)$ from (4.4.4) and including the boundary values 0 and $p_q^{max}(k)$, the optimum power allocation for the problem (P1) is the following waterfilling solution:

$$p_q^*(k) = \left[\mu_q - \sigma_q^2(k) - \sum_{r \neq q} F_{rq}(k) p_r(k) - \epsilon_q \sqrt{\sum_{r \neq q} p_r^2(k)}\right]_0^{p_q^{max}(k)} \tag{4.A.9}$$

□

Appendix 4.B Proof of Lemma 4.1

Given the waterfilling mapping $\mathrm{RWF}^{\mathrm{S}}(\cdot)$ defined as

$$\mathrm{RWF}^{\mathrm{S}}(\mathbf{p}) = (\mathrm{RWF}^{\mathrm{S}}_{\mathrm{q}}(\mathbf{p}_{-q}))_{q\in\Omega} \;:\; \mathcal{P} \mapsto \mathcal{P}, \tag{4.B.1}$$

where $\mathcal{P} \triangleq \mathcal{P}_1 \times \cdots \times \mathcal{P}_Q$, with $\mathcal{P}_q$ and $\mathrm{RWF}^{\mathrm{S}}_{\mathrm{q}}(\mathbf{p}_{-q})$ respectively defined in (4.3.2) and (4.4.6), the block-maximum norm is defined as [62]

$$\|\,\mathrm{RWF}^{\mathrm{S}}(\mathbf{p})\|^{\mathbf{w}}_{2,\mathrm{block}} \triangleq \max_{q\in\Omega} \frac{\|\,\mathrm{RWF}^{\mathrm{S}}_{\mathrm{q}}(\mathbf{p}_q)\|_2}{w_q}, \tag{4.B.2}$$

where $\mathbf{w} \triangleq [w_1, \ldots, w_Q]^T > \mathbf{0}$ is any positive weight vector. The vector weighted maximum norm is given by [107]

$$||\mathbf{x}||^{\mathbf{w}}_{\infty,\mathrm{vec}} \triangleq \max_{q\in\Omega} \frac{|x_q|}{w_q}, \qquad \mathbf{w} > \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^Q. \tag{4.B.3}$$

The matrix weighted maximum norm is given by [107]

$$||\mathbf{A}||^{\mathbf{w}}_{\infty,\mathrm{mat}} \triangleq \max_{q} \frac{1}{w_q} \sum_{r=1}^{Q} |[\mathbf{A}]_{qr}| w_r, \qquad \mathbf{A} \in \mathbb{R}^{Q\times Q}. \tag{4.B.4}$$

The mapping $\mathrm{RWF}^{\mathrm{S}}(\cdot)$ is said to be a block-contraction[4] with modulus α with respect to the norm $\|\cdot\|^{\mathbf{w}}_{2,\mathrm{block}}$ if there exists $\alpha \in [0, 1)$ such that,

[4]The mapping $\mathbf{T}$ is called a block-contraction with modulus $\alpha \in [0, 1)$ if it is a contraction in the block-maximum norm with modulus α [62].

$\forall \mathbf{p}^{(1)}, \mathbf{p}^{(2)} \in \mathcal{P}$,

$$\| \, \mathrm{RWF}^{\mathrm{S}}(\mathbf{p}^{(1)}) - \mathrm{RWF}^{\mathrm{S}}(\mathbf{p}^{(2)}) \|_{2,\mathrm{block}}^{\mathbf{w}} \leq \alpha \| \mathbf{p}^{(1)} - \mathbf{p}^{(2)} \|_{2,\mathrm{block}}^{\mathbf{w}}, \tag{4.B.5}$$

where $\mathbf{p}^{(i)} = \left(\mathbf{p}_q^{(i)}, \ldots, \mathbf{p}_q^{(i)}\right)$ for $i = 1, 2$.

Given $\mathbf{f}_q^{(i)} = \left[\sqrt{\sum_{r \neq q} p_r^2(1)^{(i)}}, \ldots, \sqrt{\sum_{r \neq q} p_r^2(N)^{(i)}}\right]^T$ for $i = 1, 2$, for each $q \in \Omega$, let $\Delta \mathbf{f}_q \triangleq \left\| \mathbf{f}_q^{(1)} - \mathbf{f}_q^{(2)} \right\|_2$ and $\mathbf{p}_{-q}(k)^{(i)} \triangleq [p_1(k)^{(i)}, \ldots, p_{q-1}(k)^{(i)}, p_{q+1}(k)^{(i)}, \ldots, p_Q(k)^{(i)}]$. Then,

$$\begin{aligned}
\Delta \mathbf{f}_q &= \left[\sum_{k=1}^{N} \left(\sqrt{\sum_{r \neq q} p_r^2(k)^{(1)}} - \sqrt{\sum_{r \neq q} p_r^2(k)^{(2)}} \right)^2 \right]^{\frac{1}{2}}, && (4.B.6) \\
&= \left[\sum_{k=1}^{N} \left(\| \mathbf{p}_{-q}(k)^{(1)} \|_2 - \| \mathbf{p}_{-q}(k)^{(2)} \|_2 \right)^2 \right]^{\frac{1}{2}}, && (4.B.7) \\
&\leq \left[\sum_{k=1}^{N} \left\| \mathbf{p}_{-q}(k)^{(1)} - \mathbf{p}_{-q}(k)^{(2)} \right\|_2^2 \right]^{\frac{1}{2}}, && (4.B.8) \\
&= \left[\sum_{k=1}^{N} \sum_{r \neq q} \left(p_r^2(k)^{(1)} + p_r^2(k)^{(2)} - 2 p_r(k)^{(1)} p_r(k)^{(2)} \right) \right]^{\frac{1}{2}}, && (4.B.9) \\
&= \left[\sum_{r \neq q} \left\| \mathbf{p}_r^{(1)} - \mathbf{p}_r^{(2)} \right\|_2^2 \right]^{\frac{1}{2}}. && (4.B.10)
\end{aligned}$$

where (4.B.8) follows from [107, Lemma 5.1.2]. Now, define for each $q \in \Omega$,

$$\begin{aligned}
e_{\mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}}} &\triangleq \left\| \mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}} \left(\mathbf{p}_{-q}^{(1)} \right) - \mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}} \left(\mathbf{p}_{-q}^{(2)} \right) \right\|_2, && (4.B.11) \\
e_q &\triangleq \left\| \mathbf{p}_q^{(1)} - \mathbf{p}_q^{(2)} \right\|_2. && (4.B.12)
\end{aligned}$$

Then, using (4.4.6) in (4.B.11), $e_{\mathrm{RWF}_\mathrm{q}^\mathrm{S}}$ can be written as

$$\begin{aligned}
e_{\mathrm{RWF}_\mathrm{q}^\mathrm{S}} &= \Big\| \Big[-\boldsymbol{\sigma}_q - \textstyle\sum_{r\neq q} \mathbf{F}_{rq}\mathbf{p}_r^{(1)} - \epsilon_q \mathbf{f}_q^{(1)} \Big]_{\mathcal{P}_q} \\
&\qquad - \Big[-\boldsymbol{\sigma}_q - \textstyle\sum_{r\neq q} \mathbf{F}_{rq}\mathbf{p}_r^{(2)} - \epsilon_q \mathbf{f}_q^{(2)} \Big]_{\mathcal{P}_q} \Big\|_2, && (4.B.13)\\
&\leq \Big\| \textstyle\sum_{r\neq q} \mathbf{F}_{rq}\mathbf{p}_r^{(1)} + \epsilon_q \mathbf{f}_q^{(1)} - \textstyle\sum_{r\neq q} \mathbf{F}_{rq}\mathbf{p}_r^{(2)} - \epsilon_q \mathbf{f}_q^{(2)} \Big\|_2, && (4.B.14)\\
&= \Big\| \sum_{r\neq q} \mathbf{F}_{rq} \left(\mathbf{p}_r^{(1)} - \mathbf{p}_r^{(2)}\right) + \epsilon_q \left(\mathbf{f}_q^{(1)} - \mathbf{f}_q^{(2)}\right) \Big\|_2, && (4.B.15)\\
&\leq \Big\| \sum_{r\neq q} \mathbf{F}_{rq} \left(\mathbf{p}_r^{(1)} - \mathbf{p}_r^{(2)}\right) \Big\|_2 + \epsilon_q \Big\| \mathbf{f}_q^{(1)} - \mathbf{f}_q^{(2)} \Big\|_2, && (4.B.16)\\
&\leq \Big\| \sum_{r\neq q} \mathbf{F}_{rq} \left(\mathbf{p}_r^{(1)} - \mathbf{p}_r^{(2)}\right) \Big\|_2 + \epsilon_q \left[\sum_{r\neq q} \Big\| \mathbf{p}_r^{(1)} - \mathbf{p}_r^{(2)} \Big\|_2^2 \right]^{\frac{1}{2}} && (4.B.17)\\
&\leq \sum_{r\neq q} \left(\max_k \; F_{rq}(k) \right) \Big\| \mathbf{p}_r^{(1)} - \mathbf{p}_r^{(2)} \Big\|_2 \\
&\qquad + \epsilon_q \left[\sum_{r\neq q} \Big\| \mathbf{p}_r^{(1)} - \mathbf{p}_r^{(2)} \Big\|_2^2 \right]^{\frac{1}{2}}, && (4.B.18)\\
&= \sum_{r\neq q} \left(\max_{k\in\mathcal{D}_q\cap\mathcal{D}_r} F_{rq}(k) \right) e_r + \epsilon_q \Big[\sum_{r\neq q} e_r^2 \Big]^{\frac{1}{2}}, && (4.B.19)\\
&\leq \sum_{r\neq q} \Big([\mathbf{S}^{max}]_{rq} + \epsilon_q \Big) e_r, && (4.B.20)
\end{aligned}$$

$\forall\ \mathbf{p}_q^{(1)}, \mathbf{p}_q^{(2)} \in \mathcal{P}_q$ and $\forall\ q \in \Omega$, where: (4.B.14) follows from the nonexpansive property of the waterfilling projection [92, Lemma 3]; (4.B.16) follows from the triangle inequality [107]; (4.B.17) follows from (4.B.10); (4.B.18) and (4.B.19) follow from the definitions of $\mathbf{F}_{rq}$ and e_q respectively from (4.4.7) and (4.B.12); and (4.B.20) follows from the definition of $\mathbf{S}^{max}$ in (4.4.12) and Jensen's inequality [51].

The set of inequalities in (4.B.20) can be written in vector form as

$$\mathbf{0} \leq \mathbf{e}_{\mathrm{RWF^S}} \leq (\mathbf{S}^{max} + \mathbf{E})\mathbf{e}, \tag{4.B.21}$$

where $\mathbf{E}$ is defined in (4.4.11) and the vectors $\mathbf{e}_{\mathrm{RWF^S}}$ and $\mathbf{e}$ are defined as

$$\mathbf{e}_{\mathrm{RWF^S}} \triangleq \left[e_{\mathrm{RWF^S}_1}, \ldots, e_{\mathrm{RWF^S}_Q}\right]^T, \text{ and } \mathbf{e} = [e_1 \ldots, e_Q]^T. \tag{4.B.22}$$

Using the vector and matrix weighted maximum norms from (4.B.3) and (4.B.4) respectively, (4.B.21) can be written as

$$\begin{aligned} \|\mathbf{e}_{\mathrm{RWF^S}}\|^{\mathbf{w}}_{\infty,\mathrm{vec}} &\leq \|(\mathbf{S}^{max} + \mathbf{E})\mathbf{e}\|^{\mathbf{w}}_{\infty,\mathrm{vec}}, && (4.B.23)\\ &\leq \|\mathbf{S}^{max} + \mathbf{E}\|^{\mathbf{w}}_{\infty,\mathrm{mat}} \cdot \|\mathbf{e}\|^{\mathbf{w}}_{\infty,\mathrm{vec}}, && (4.B.24) \end{aligned}$$

$\forall\ \mathbf{w} > \mathbf{0}$. Using the block-maximum norm (4.B.2),

$$\begin{aligned} \|\mathbf{e}_{\mathrm{RWF^S}}\|^{\mathbf{w}}_{\infty,\mathrm{vec}} &= \|\,\mathrm{RWF^S}\,(\mathbf{p}^{(1)}) - \mathrm{RWF^S}\,(\mathbf{p}^{(2)})\|^{\mathbf{w}}_{2,\mathrm{block}}, && (4.B.25)\\ &\leq \|\mathbf{S}^{max} + \mathbf{E}\|^{\mathbf{w}}_{\infty,\mathrm{mat}} \left\|\mathbf{p}^{(1)}_r - \mathbf{p}^{(2)}_r\right\|^{\mathbf{w}}_{2,\mathrm{block}}, && (4.B.26) \end{aligned}$$

$\forall \mathbf{p}^{(2)}, \mathbf{p}^{(2)} \in \mathcal{P}$, with $\mathbf{E}$ and $\mathbf{S}$ as defined in (4.4.11) and (4.4.12) respectively. It is clear that $\mathrm{RWF^S}(\cdot)$ is a block contraction when $||\mathbf{S}^{max} + \mathbf{E}||^{\mathbf{w}}_{\infty,\mathrm{mat}} < 1$. □

Appendix 4.C Proof of Theorem 4.2

From [108], every concave game[5] has at least one equilibrium. For the game $\mathscr{G}_{\text{rob}}^{S}$:

1. The set of feasible strategy profiles, $\mathcal{P}_q$ of each player q is compact and convex.

2. The payoff function of each player q in (4.3.6) is continuous in $\mathbf{p} \in \mathcal{P}$ and concave in $\mathbf{p}_q \in \mathcal{P}_q$.

Thus, the game $\mathscr{G}_{\text{rob}}^{S}$ has at least one equilibrium. From Lemma 4.1, the waterfilling mapping $\text{RWF}^{\text{S}}(\cdot)$ is a block-contraction if (4.4.14) is satisfied for some $\mathbf{w} > \mathbf{0}$. Thus, the robust-optimization equilibrium of game $\mathscr{G}_{\text{rob}}^{S}$ is unique (using [63, Theorem 1]). Since $\mathbf{S}^{max} + \mathbf{E}$ is a nonnegative matrix, there exists a positive vector $\bar{\mathbf{w}}$ such that

$$||\mathbf{S}^{max} + \mathbf{E}||_{\infty,\text{mat}}^{\bar{\mathbf{w}}} < 1 \tag{4.C.1}$$

Using [62, Corollary 6.1] and the triangle inequality [107], this is satisfied when

$$||\mathbf{S}^{max}||_{\infty,\text{mat}}^{\bar{\mathbf{w}}} + ||\mathbf{E}||_{\infty,\text{mat}}^{\bar{\mathbf{w}}} < 1 \;\Rightarrow\; \rho(\mathbf{S}^{max}) < 1 - \rho(\mathbf{E}). \tag{4.C.2}$$

□

[5] A game is said to be concave if the payoff functions are concave and the sets of admissible strategies are compact and convex.

Appendix 4.D Proof of Theorem 4.3

From Lemma 4.1 and (4.C.1) the waterfilling mapping $\mathrm{RWF}^{\mathrm{S}}(\cdot)$ is a block-contraction. From [63, Theorem 2], the robust iterative waterfilling algorithm described in Algorithm 4.1 converges to the unique robust-optimization equilibrium of game $\mathscr{G}_{\mathrm{rob}}^{S}$ for any set of feasible initial conditions and any update schedule.

Chapter 5

SUM-RATE ANALYSIS IN THE TWO-USER SCENARIO

In this chapter, the effect of uncertainty on the sum-rate and efficiency of the system for the two-user scenario in the game $\mathscr{G}_{\text{rob}}^{S}$ presented in the previous chapter is analysed (Refer to Section 2.3 for a discussion on equilibrium efficiency). The sum-rate of the system, S, is given by

$$S = \sum_{q=1}^{Q} R_q. \tag{5.0.1}$$

where R_q is the information rate of user q as defined in (4.2.5). In game $\mathscr{G}_{\text{rob}}^{S}$, the price of stability and anarchy are the same under the sufficient conditions in Theorem 4.2 due to the existence of a unique equilibrium. Thus, the price of anarchy, PoA, defined as the ratio of the sum-rate of the system at the social optimal solution, S^*, and the sum-rate of the system at the robust-optimization equilibrium, S^{rob}, is given by

$$\text{PoA} = \frac{S^*}{S^{\text{rob}}}. \tag{5.0.2}$$

Note that a lower price of anarchy indicates that the robust-optimization equilibrium is more efficient.

For the two user case in the game $\mathscr{G}^S_{\text{rob}}$, the worst-case interference in each frequency reduces to $(F_{rq}(k) + \epsilon_q)p_r(k)$ with $q, r = 1, 2$ and $q \neq r$. This means that the robust waterfilling operation for the two user case ($Q = 2$) is simply the standard waterfilling solution with the worst-case channel coefficients. For the sake of clarity, the analysis here is restricted to both users having identical noise variance $\sigma_1^2(k) = \sigma_2^2(k) = \sigma^2 \ \forall k$ across all frequencies, identical uncertainty bounds $\epsilon_1 = \epsilon_2 = \epsilon$ and identical total power constraints $\sum_{k=1}^N p_1(k) = \sum_{k=1}^N p_2(k) = P_T$. The results presented here can be extended to the non-identical case along similar lines. In order to develop a clear understanding of the the behaviour of the equilibrium, the sum-rate of the system is first analyzed for a system with two frequencies ($N = 2$) and then extended to systems with large ($N \to \infty$) number of frequencies.

5.1 Two frequency case ($N = 2$)

Consider a two-frequency anti-symmetric system as shown in Figure 5.1 where the channel gains are $|H_{11}(1)|^2 = |H_{11}(2)|^2 = 1$, $|H_{22}(1)|^2 = |H_{22}(2)|^2 = 1$, $|H_{12}(2)|^2 = |H_{21}(1)|^2 = \alpha$ and $|H_{12}(1)|^2 = |H_{21}(2)|^2 = m\alpha$ with $m > 1$

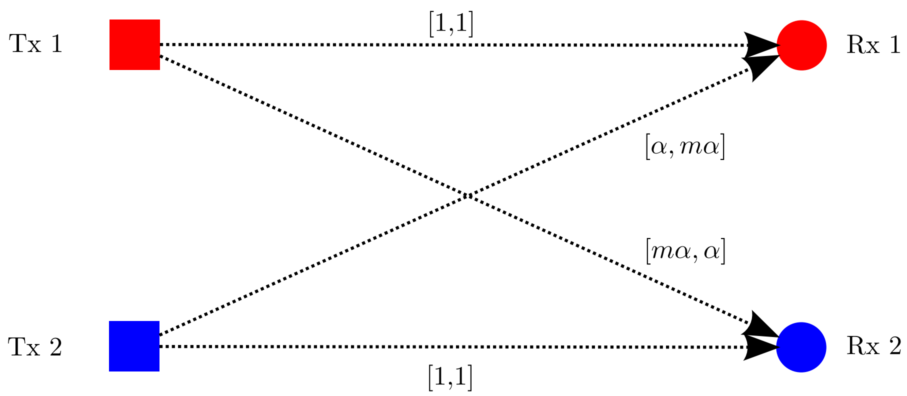


Figure 5.1: Anti-symmetric system with $Q = 2$, $N = 2, \epsilon_1 = \epsilon_2 = \epsilon$. The noise variances for both users in both frequencies are σ^2. The channel gains are: $|H_{11}(1)|^2 = |H_{11}(2)|^2 = |H_{22}(1)|^2 = |H_{22}(2)|^2 = 1; |H_{12}(2)|^2 = |H_{21}(1)|^2 = \alpha$ and $|H_{12}(1)|^2 = |H_{21}(2)|^2 = m\alpha$ with $m > 1$ and $0 < \alpha < 1$. The power allocations for this system at the robust-optimization equilibrium are presented in (5.1.1).

and $0 < \alpha < 1$. The total power constraint for the two users is $p_1(1)+p_1(2) = 1$ and $p_2(1)+p_2(2) = 1$. From (4.4.3) the power allocations at the equilibrium are,

$$\begin{aligned} p_1(1) &= (\mu_1 - \sigma^2 - (\alpha + \epsilon)p_2(1))^+, \\ p_1(2) &= (\mu_1 - \sigma^2 - (m\alpha + \epsilon)p_2(2))^+, \\ p_2(1) &= (\mu_2 - \sigma^2 - (m\alpha + \epsilon)p_1(1))^+, \\ p_2(2) &= (\mu_2 - \sigma^2 - (\alpha + \epsilon)p_1(2))^+. \end{aligned} \tag{5.1.1}$$

The following theorem presents the effect of uncertainty on the sum-rate and price of anarchy of the system for the high interference and low interference cases:

Theorem 5.1. *For the two-user two-frequency anti-symmetric system described in Figure 5.1:*

- *High interference: When $\sigma^2 \ll \alpha(1-p)$, the sum-rate increases and the price of anarchy decreases as the channel uncertainty bound increases.*
- *Low interference: When $\sigma^2 \gg m\alpha p$, the sum-rate decreases and the price of anarchy increases as the channel uncertainty bound increases.*

Proof. See Appendix 5.A. □

From this result, it is evident that the robust-optimization equilibrium behaves in opposite ways when there is high interference and when there is low interference in the system. This suggests that there might be a certain level of interference where the sum-rate and price of anarchy do not change with change in uncertainty. This is given by the following proposition:

Proposition 5.1. *At the level of interference $\alpha = \alpha_o$, where*

$$\alpha_o = \frac{\sigma^2}{2m}\left(\left((m+1)^2 + \frac{4m}{\sigma^2}\right)^{\frac{1}{2}} - m - 1\right), \tag{5.1.2}$$

the sum-rate and the price of anarchy are independent of the level of uncertainty or the power allocation used. Furthermore, at this value of interference, the price of anarchy is equal to unity.

Proof. See Appendix 5.B □

It can be seen that even for such a simple system, the global behaviour of the robust-optimization equilibrium appears to be quite complex. This indicates that the global properties of the robust-optimization equilibrium for larger systems is quite strongly dependent on the level of interference in the system, which is seen in the subsequent results for systems with large number of frequencies. However, the underlying behaviour of the equilibrium for the two-frequency case is that the equilibrium moves towards the frequency division multiple access (FDMA) solution as the uncertainty bound increases (from (5.A.2)). This helps in providing a way to analyze the equilibrium behaviour for systems with large number of frequencies in the following section.

5.2 Large number of frequencies ($N \to \infty$)

In this section, the equilibrium behaviour is characterized for the two-user system with a large number of frequencies. As seen from the previous section, the equilibrium tends to move towards the FDMA solution as uncertainty bound increases. In order to quantify this effect, the quantity $J(k)$, defined as

$$J(k) \triangleq -p_1(k)p_2(k), \tag{5.2.1}$$

is used as a measure of the extent of partitioning of the frequency k between the two users. It is minimum ($J(k) = -1$) when both the users allocate all their total power to the same frequency k and is maximum ($J(k) = 0$) when at most one user is occupying the frequency k. Note that $J(k) = 0 \ \forall k \in \{1, \ldots, N\}$ when the users adopt an FDMA scheme.

The following lemma describes the effect of the uncertainty bound on the extent of partitioning of the system:

Lemma 5.1. *For the two-user case in the game* $\mathscr{G}^S_{\text{rob}}$*, when the number of frequencies,* $N \to \infty$*, the extent of partitioning in every frequency is non-decreasing as the uncertainty bound of the system increases for any set of channel values, i.e.,*

$$\frac{\partial}{\partial \epsilon} J(k) \geq 0 \quad \forall k \in \{1, \ldots, N\} \quad \textit{when} \quad N \to \infty, \tag{5.2.2}$$

with equality for frequencies where $J(k) = 0$*, where* $J(k)$ *is defined in* (5.2.1).

Proof. See Appendix 5.C. □

The above lemma suggests that the robust-optimization equilibrium moves towards greater frequency-space partitioning as the uncertainty bound increases when there is a large number of frequencies in the system. In other words, the equilibrium is moving closer to an FDMA solution under increased channel uncertainty. When the FDMA solution is globally (sum-rate) optimal, this will lead to an improvement in the sum-rate at the equilibrium. This is stated in the following theorem:

Theorem 5.2. *For the two-user case in the game* $\mathscr{G}^S_{\text{rob}}$*, as the number of frequencies,* $N \to \infty$*, the sum-rate (price of anarchy) at the robust-optimization equilibrium of the system is non-decreasing (non-increasing) as the uncertainty bound increases if,* $\forall\ k \in \{1, \ldots, N\}$*,*

$$\big(F_{21}(k) - \epsilon\big)\big(F_{12}(k) - \epsilon\big) > \frac{1}{4}. \tag{5.2.3}$$

Proof. See Appendix 5.D. □

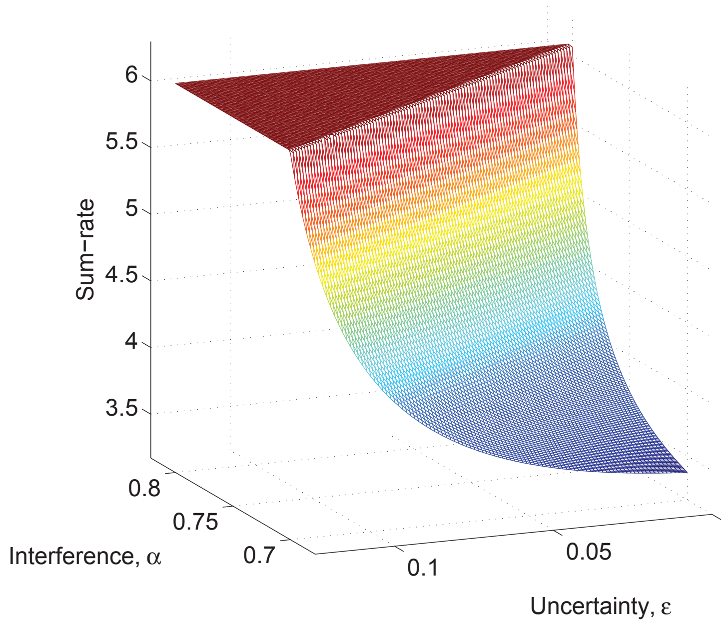
For the special case of frequency-flat systems, at the equilibrium, all users have equal power allocation to all frequencies. This solution is not dependent on the uncertainty in the CSI. This leads to no change in the extent of partitioning and thus sum-rate and price of anarchy are not affected by uncertainty.

Also, the results of this section are not just limited to the robust-optimization equilibrium for the system presented here. When the uncertainty $\epsilon = 0$, the framework presented here can be used to analyse the behaviour of the Nash equilibrium of the iterative waterfilling algorithm as a function of the interference coefficients.

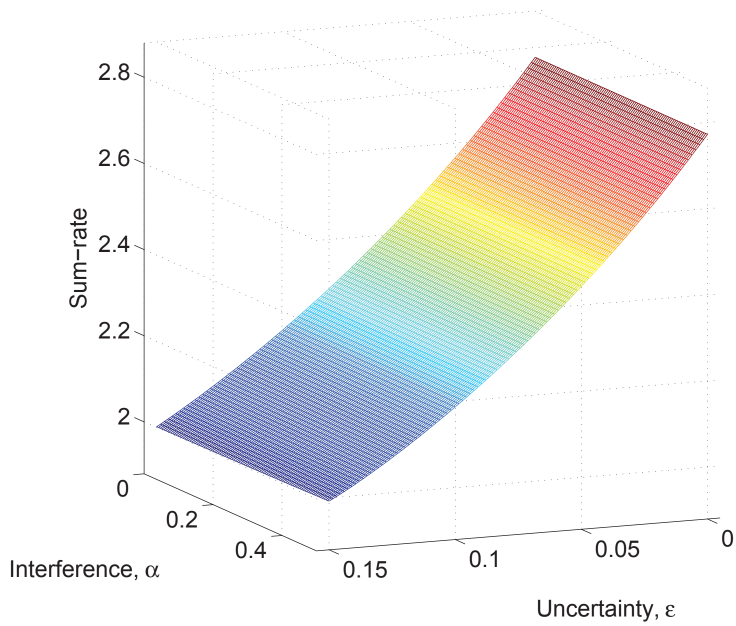
5.3 Simulation results

In this section, some simulation results to study the impact of channel uncertainty on the equilibrium are presented. Figure 5.2 shows the simulation results for the two user and two frequency scenario and Figure 5.3 shows the results for a two user case with $N = 8$ frequencies.

The sum-rate of the system in Figure 5.1 under high interference conditions is plotted as a function of interference and uncertainty in Figure 5.2a. The flat region corresponds to the sum-rate at Pareto optimal solution (FDMA) and the edge of the surface corresponds to the sufficient condition in (4.5.2). It can be seen that the Nash equilibrium (when the uncertainty bound $\epsilon = 0$) moves closer to the Pareto optimal solution as the interference increases. It is also evident that the sum-rate increases for a fixed interfer-

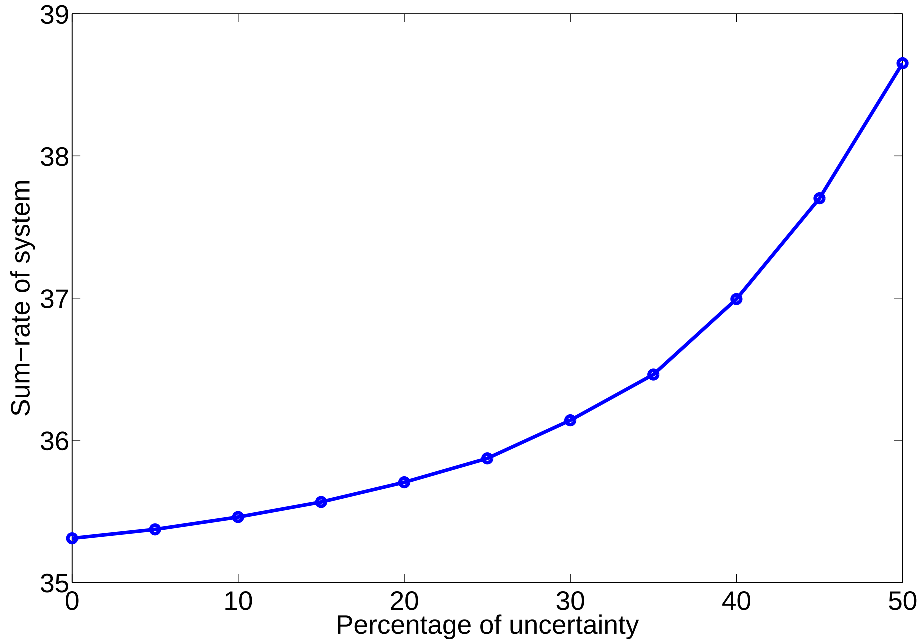


(a) Sum-rate under high interference vs. uncertainty and interference.

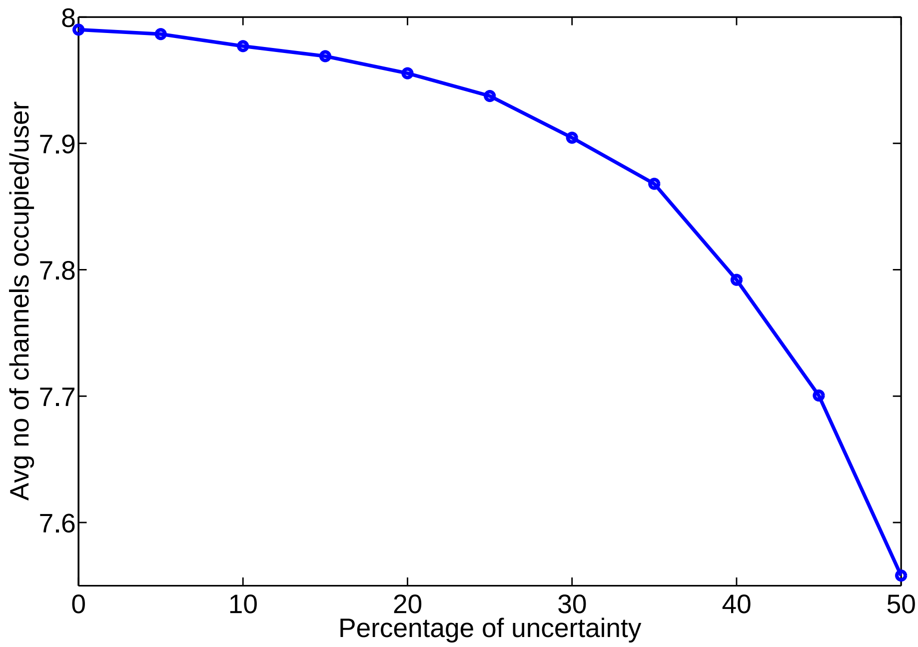


(b) Sum-rate under low interference vs. uncertainty and interference.

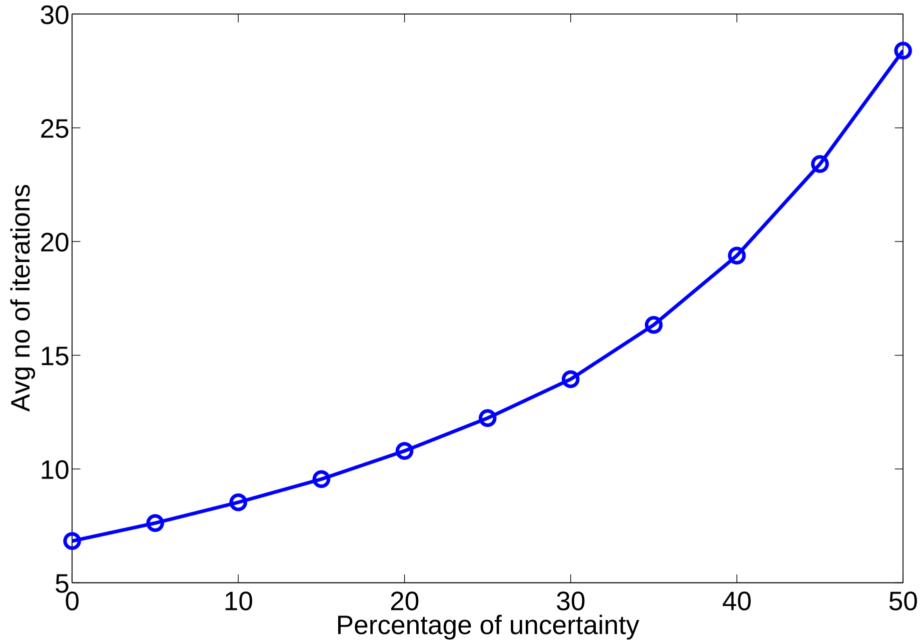
Figure 5.2: Simulation results for the anti-symmetric system in Figure 5.1. Note that the zero uncertainty corresponds to the Nash equilibrium



(a) Sum-rate of the system vs. uncertainty δ.



(b) Average number of channels occupied per user vs. uncertainty δ.



(c) Average number of iterations vs. uncertainty δ.

Figure 5.3: Simulation results demonstrating the effect of channel uncertainty on (a) Sum-rate, (b) Average number of channels/user, (c) Average number of iterations for a system with $\sigma = 0.1$, $Q = 2$ users and $N = 8$ frequencies averaged over 1000 runs. Channel gains $H_{rq}(k) \sim N_C(0,1)$ for $r \neq q$, $H_{qq}(k) \sim N_C(0,4)$. Channel uncertainty model: nominal value $F_{rq}(k) = F_{rq}^{\text{true}}(k)(1 + e_{rq}(k))$ with $e_{rq}(k) \sim U(-\frac{\delta}{2}, \frac{\delta}{2})$, $\delta < 1$. The simulations are limited to channels which satisfy the sufficiency condition in (4.5.2). Note that the zero uncertainty solution corresponds to the Nash equilibrium.

ence as uncertainty increases, as expected from Theorem 5.1. In Figure 5.2b, the sum-rate at low interference is plotted as a function of interference and uncertainty. As expected from Theorem 5.1, the sum-rate decreases as the uncertainty bound increases. Simulation results similar to those in Section 4.7 for a two-user system with 8 frequencies are presented in Figure 5.3.

As seen before, improvement in sum-rate due to lower channel occupancy leads to a trade-off in increased number of iterations to convergence.

From these simulations it can be seen that the analytical results derived in Section 5.2 for very large number of frequencies also hold true for a finite number of frequencies. From the simulation results and the theoretical analysis, it can be concluded that the robust-optimization equilibrium moves towards an FDMA solution as the uncertainty bound increases.

5.4 Summary

In this chapter, the efficiency of the equilibrium of the robust rate-maximization game proposed in Chapter 4 was investigated for the two-user scenario. In order to develop a clearer understanding of the behaviour of the equilibrium, a simple two-frequency system was first analyzed. The effect of uncertainty on the sum-rate and the price of anarchy of this two-frequency system were derived for two regimes, viz. high interference and low interference. Following this, the effect of uncertainty on the sum-rate and price of anarchy of a two-user system with asymptotically large number of frequencies were characterized. Finally, these effects were demonstrated through simulation results.

Appendix 5.A Proof of Theorem 5.1

Let $p_1(1) = p$. Hence, by symmetry, $p_1(2) = p_2(1) = 1 - p$, $p_2(2) = p$ and $\mu_1 = \mu_2 = \mu$. Consider the interior operating points of the robust waterfilling operator $\mathrm{RWF}_{\mathrm{q}}^{\mathrm{S}}(\cdot)$ where it is linear. Eliminating μ from (5.1.1),

$$p = \frac{1-\alpha-\epsilon}{2(1-(m+1)\frac{\alpha}{2}-\epsilon)} \geq 0.5. \tag{5.A.1}$$

The gradient of p with respect to ϵ is

$$\frac{\partial p}{\partial \epsilon} = \frac{(m-1)\alpha}{4(1-(m+1)\frac{\alpha}{2}-\epsilon)^2} > 0. \tag{5.A.2}$$

Thus, the robust-optimization equilibrium moves towards the FDMA solution as the uncertainty bound increases.

The signal-to-interference-plus-noise ratio (SINR) for the two users in the two frequency bins is given by

$$\mathrm{SINR}_1(1) = \mathrm{SINR}_2(2) = \frac{p}{\sigma^2+\alpha(1-p)}, \tag{5.A.3}$$
$$\mathrm{SINR}_1(2) = \mathrm{SINR}_2(1) = \frac{1-p}{\sigma^2+m\alpha p}\ . \tag{5.A.4}$$

and the sum-rate of the system is

$$S \triangleq 2\log\Big(1+\frac{p}{\sigma^2+\alpha(1-p)}\Big)+2\log\Big(1+\frac{1-p}{\sigma^2+m\alpha p}\Big). \tag{5.A.5}$$

Case 1: High interference scenario

In the high interference scenario, $\sigma^2 \ll \alpha(1-p)$. Let $\xi = p/\alpha(1-p)$. Then, the SINR for the two users in the two frequency bins can be approximated as

$$\begin{aligned} \mathrm{SINR}_1(1) = \mathrm{SINR}_2(2) &\approx \frac{p}{\alpha(1-p)} = \xi \ , \\ \mathrm{SINR}_1(2) = \mathrm{SINR}_2(1) &\approx \frac{1-p}{m\alpha p} = \frac{1}{m\alpha^2\xi} \ . \end{aligned} \tag{5.A.6}$$

The sum-rate of the system at high interference can be approximated as

$$S \approx 2\log\left((1+\xi)(1+\frac{1}{m\alpha^2\xi})\right), \tag{5.A.7}$$

$$= 2\log\left(1+\frac{1}{m\alpha^2}+\xi+\frac{1}{m\alpha^2\xi}\right). \tag{5.A.8}$$

The goal here is to analyse the behaviour of the sum-rate S as the uncertainty ϵ increases. To this end, it has to be shown that the gradient of the sum-rate with respect to ϵ is positive. Since $\log(x)$ increases monotonically with x, consider

$$\begin{aligned} \frac{\partial}{\partial\epsilon}(\xi+\frac{1}{m\alpha^2\xi}) &= \left(1-\frac{1}{m\alpha^2\xi^2}\right)\frac{\partial\xi}{\partial\epsilon}, \\ &= \left(1-\frac{(1-p)^2}{mp^2}\right)\frac{\partial\xi}{\partial\epsilon}, \end{aligned} \tag{5.A.9}$$

and $(1-\frac{(1-p)^2}{mp^2}) > 0$ since $p \geq 0.5$ and $m > 1$. Now,

$$\frac{\partial\xi}{\partial\epsilon} = \frac{1}{\alpha(1-p)^2}\frac{\partial p}{\partial\epsilon} \ . \tag{5.A.10}$$

From (5.A.2), (5.A.9) and (5.A.10),

$$\frac{\partial S}{\partial\epsilon} > 0. \tag{5.A.11}$$

Thus, the sum-rate of the system increases as the uncertainty ϵ increases. This also shows that the robust-optimization equilibrium achieves a higher sum-rate in the presence of channel uncertainty ($\epsilon > 0$) than the Nash equilibrium at zero uncertainty ($\epsilon = 0$).

The social optimal solution for this system at high interference is frequency division multiplexing [23]. In other words, the frequency space is fully partitioned at the social optimal solution. The sum-rate at the social optimal solution for the given system at high interference, S^*, is given by

$$S^* = 2\log\Big(1 + \frac{1}{\sigma^2}\Big). \tag{5.A.12}$$

The price of anarchy at high interference, PoA, is

$$\mathrm{PoA} = \frac{\log\left(1 + \frac{1}{\sigma^2}\right)}{\log\left(1 + \frac{1}{m\alpha^2} + \xi + \frac{1}{m\alpha^2\xi}\right)}. \tag{5.A.13}$$

Since $\frac{\partial S}{\partial \epsilon} > 0$, it implies that $\frac{\partial \,\mathrm{PoA}}{\partial \epsilon} < 0$.

Case 2: Low interference scenario

In the low interference scenario, i.e. when $m\alpha p \ll \sigma^2$, the signal-to-interference+noise ratio SINR for the two users in the two frequency bins can be approximately written as

$$\begin{aligned} \mathrm{SINR}_1(1) = \mathrm{SINR}_2(2) &\approx \frac{p}{\sigma^2}, \\ \mathrm{SINR}_1(2) = \mathrm{SINR}_2(1) &\approx \frac{1-p}{\sigma^2}. \end{aligned} \tag{5.A.14}$$

The sum-rate of the system at low interference can be approximated as

$$S \approx 2\log\left(\left(1+\frac{p}{\sigma^2}\right)\left(1+\frac{1-p}{\sigma^2}\right)\right), \tag{5.A.15}$$

$$= 2\log\left(1+\frac{1}{\sigma^2}+\frac{p-p^2}{\sigma^2}\right). \tag{5.A.16}$$

Now,

$$\frac{\partial S}{\partial \epsilon} = \left(1+\frac{1}{\sigma^2}+\frac{p-p^2}{\sigma^2}\right)^{-1} \frac{(1-2p)}{\sigma^2}\frac{\partial p}{\partial \epsilon} < 0. \tag{5.A.17}$$

At low interference, the system behaves similar to a parallel Gaussian channel system. The social optimal solution in this scenario is the waterfilling solution and leads to equal power allocation to both bins. The sum-rate at the social optimal solution for the given system at low interference, S^*, is given by

$$S^* = 4\log\left(1+\frac{1}{2\sigma^2}\right). \tag{5.A.18}$$

The price of anarchy at low interference, PoA, is

$$\text{PoA} = \frac{4\log\left(1+\frac{1}{2\sigma^2}\right)}{2\log\left(1+\frac{1}{\sigma^2}+\frac{p-p^2}{\sigma^2}\right)} = \frac{\log\left(1+\frac{1}{\sigma^2}+\frac{1}{4\sigma^4}\right)}{\log\left(1+\frac{1}{\sigma^2}+\frac{p-p^2}{\sigma^2}\right)}. \tag{5.A.19}$$

Note that, at low interference, $m\alpha p \ll 1$. From (5.A.1), $p \approx 0.5$. Thus the PoA is close to unity. Since $\frac{\partial S}{\partial \epsilon} < 0$, $\frac{\partial}{\partial \epsilon}\text{PoA} > 0$. □

Appendix 5.B Proof of Proposition 5.1

The gradient of the sum-rate S_{rob} with respect to ϵ is

$$\frac{\partial S_{rob}}{\partial \epsilon} = \frac{\partial S_{\text{rob}}}{\partial p}\frac{\partial p}{\partial \epsilon}. \tag{5.B.1}$$

From (5.A.2), we have $\frac{\partial p}{\partial \epsilon} > 0$. Now,

$$\frac{\partial S_{\text{rob}}}{\partial p} = 2\frac{\frac{1}{\sigma^2+\alpha(1-p)} + \frac{\alpha p}{(\sigma^2+\alpha(1-p))^2}}{1+\frac{p}{\sigma^2+\alpha(1-p)}} - 2\frac{\frac{1}{\sigma^2+m\alpha p} + \frac{(1-p)m\alpha}{(\sigma^2+m\alpha p)^2}}{1+\frac{1-p}{\sigma^2+m\alpha p}}. \tag{5.B.2}$$

Setting $\frac{\partial S_{\text{rob}}}{\partial p} = 0$, we solve for α to get the following roots,

$$\alpha = \begin{cases} 0, \\ \frac{-\sigma^2}{2m}\left(m+1 \pm \left(4m/\sigma^2 + (m+1)^2\right)^{\frac{1}{2}}\right), \\ \frac{\sigma^2(2p-1)}{(m-1)p^2+2p-1}. \end{cases} \tag{5.B.3}$$

The positive root that is independent of ϵ and p (which is a function of the uncertainty ϵ, from (5.A.1)) is the required solution where the sum-rate is constant regardless of uncertainty. Thus, the required interference value is given by

$$\alpha_{\text{o}} = \frac{\sigma^2}{2m}\left(\left(4m/\sigma^2 + (m+1)^2\right)^{\frac{1}{2}} - m - 1\right). \tag{5.B.4}$$

Since the root α_{o} of $\frac{\partial S_{\text{rob}}}{\partial p}$ is independent of p, different power allocation schemes (resulting in different values of p) will result in the same sum-rate at $\alpha = \alpha_{\text{o}}$. Thus, the price of anarchy at $\alpha = \alpha_{\text{o}}$ is unity. □

Appendix 5.C Proof of Lemma 5.1

From (4.4.3), the power allocations for the two users at the robust-optimization equilibrium in the kth frequency are

$$p_1(k) = \Big(\mu_1 - \sigma^2 - (F_{21} + \epsilon)p_2(k)\Big)^+, \tag{5.C.1}$$

$$p_2(k) = \Big(\mu_2 - \sigma^2 - (F_{12} + \epsilon)p_1(k)\Big)^+, \tag{5.C.2}$$

with $\sum_{k=1}^{N} p_1(k) = \sum_{k=1}^{N} p_2(k) = P_T$.

Let $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_{ol}$ be the sets of frequencies exclusively used by user 1, user 2 and by both respectively and $n_1 \triangleq |\mathcal{D}_1|$ and $n_2 \triangleq |\mathcal{D}_2|$ be the number of frequencies exclusively used by user 1 and user 2 respectively at the equilibrium. Then, from (5.C.1), $p_1(k) = \mu_1 - \sigma_1^2$ and $p_2(k) = 0 \; \forall \; k \in \mathcal{D}_1$ and $p_1(k) = 0$ and $p_2(k) = \mu_2 - \sigma_1^2 \; \forall \; k \in \mathcal{D}_2$. The power remaining for allocation to the frequencies in $\mathcal{D}_{ol} \triangleq \{k_1, \ldots, k_{ol}\}$ by user 1 and user 2 is $(1 - n_1(\mu_1 - \sigma^2))$ and $(1 - n_2(\mu_2 - \sigma^2))$ respectively.

This separation of the frequency-space into exclusive-use and overlapped-use frequencies allows us to analyse the system without the nonlinear operation $(\cdot)^+$. Thus, the power allocations at the fixed point in the overlapped-use frequency-space can be expressed as a system of linear equations,

$$p_1(k) + (F_{21}(k) + \epsilon)p_2(k) - \mu_1 - \sigma^2 = 0, \quad k \in \mathcal{D}_{ol} \tag{5.C.3}$$

$$(F_{12}(k) + \epsilon)p_1(k) + p_2(k) - \mu_2 - \sigma^2 = 0, \quad k \in \mathcal{D}_{ol} \tag{5.C.4}$$

$$\textstyle\sum_{k \in \mathcal{D}_{ol}} p_1(k) + n_1(\mu_1 - \sigma^2) = P_T, \tag{5.C.5}$$

$$\textstyle\sum_{k\in\mathcal{D}_{ol}} p_2(k) + n_2(\mu_2 - \sigma^2) \; = P_T. \tag{5.C.6}$$

Writing these in matrix form,

$$\begin{bmatrix} \mathbf{A}_{k_1} & & \mathbf{0} & -\mathbf{I}_2 \\ & \ddots & & \vdots \\ \mathbf{0} & & \mathbf{A}_{k_{ol}} & -\mathbf{I}_2 \\ \mathbf{I}_2 & \cdots & \mathbf{I}_2 & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{p}(k_1) \\ \vdots \\ \mathbf{p}(k_{ol}) \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_2 \\ \vdots \\ \mathbf{0}_2 \\ \mathbf{p_t} \end{bmatrix} \tag{5.C.7}$$

where

$$\begin{aligned} &\mathbf{A}_k \triangleq \begin{bmatrix} 1 & F_{21}(k)+\epsilon \\ F_{12}(k)+\epsilon & 1 \end{bmatrix}, \; \mathbf{D} \triangleq \begin{bmatrix} n_1 & 0 \\ 0 & n_2 \end{bmatrix}, \\ &\mathbf{p}(k) \triangleq \begin{bmatrix} p_1(k) \\ p_2(k) \end{bmatrix}, \quad \mathbf{p_t} \triangleq \begin{bmatrix} P_T \\ P_T \end{bmatrix} \quad \text{and} \quad \boldsymbol{\mu} \triangleq \begin{bmatrix} \mu_1 - \sigma^2 \\ \mu_2 - \sigma^2 \end{bmatrix}. \end{aligned} \tag{5.C.8}$$

Let

$$\begin{aligned} &\mathbf{A} \triangleq \begin{bmatrix} \mathbf{A}_{k_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_{k_{ol}} \end{bmatrix}, \quad \mathbf{B} \triangleq \begin{bmatrix} -\mathbf{I}_2 \\ \vdots \\ -\mathbf{I}_2 \end{bmatrix}, \\ &\mathbf{C} \triangleq [\mathbf{I}_2 \; \ldots \; \mathbf{I}_2] \quad \text{and} \quad \mathbf{P} \triangleq \begin{bmatrix} \mathbf{p}(k_1) \\ \vdots \\ \mathbf{p}(k_{ol}) \end{bmatrix}. \end{aligned} \tag{5.C.9}$$

so that (5.C.7) can be written as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{P} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{p_t} \end{bmatrix}. \tag{5.C.10}$$

This system can be solved to get

$$\begin{bmatrix} \mathbf{P} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{p_t} \end{bmatrix} = \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{p_t} \end{bmatrix} = \begin{bmatrix} \mathbf{X}\mathbf{p_t} \\ \mathbf{Z}\mathbf{p_t} \end{bmatrix}, \tag{5.C.11}$$

where

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix}. \tag{5.C.12}$$

Using [109, Fact 10.12.9] and differentiating (5.C.11) with respect to ϵ,

$$\begin{aligned} \frac{\partial}{\partial \epsilon} \begin{bmatrix} \mathbf{P} \\ \boldsymbol{\mu} \end{bmatrix} &= - \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial \epsilon}\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{p_t} \end{bmatrix}, & (5.C.13) \\ &= - \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \epsilon}\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{p_t} \end{bmatrix}, & (5.C.14) \\ &= - \begin{bmatrix} \mathbf{W}(\frac{\partial}{\partial \epsilon}\mathbf{A})\mathbf{X}\,\mathbf{p_t} \\ \mathbf{Y}(\frac{\partial}{\partial \epsilon}\mathbf{A})\mathbf{X}\,\mathbf{p_t} \end{bmatrix}. & (5.C.15) \end{aligned}$$

Due to the nature of the waterfilling function, n_1 and n_2 are non-decreasing piecewise-constant functions of ϵ. The above derivative exists only in the regions where n_1 and n_2 are constant. Using [109, Proposition 2.8.7], the partitioned matrix inverse can be written as

$$\mathbf{W} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}, \quad (5.C.16)$$

$$\mathbf{X} = -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}, \quad (5.C.17)$$

$$\mathbf{Y} = -(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}, \quad (5.C.18)$$

$$\mathbf{Z} = (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}. \quad (5.C.19)$$

Using

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_{k_1}^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_{k_{ol}}^{-1} \end{bmatrix}, \quad (5.C.20)$$

it can be shown that (the detailed derivations are provided at the end of this proof),

$$\mathbf{W} = \begin{bmatrix} \mathbf{A}_{k_1}^{-1} - \mathbf{A}_{k_1}^{-1}\mathbf{Z}\mathbf{A}_{k_1}^{-1} & \cdots & -\mathbf{A}_{k_1}^{-1}\mathbf{Z}\mathbf{A}_{k_{ol}}^{-1} \\ \vdots & & \vdots \\ -\mathbf{A}_{k_{ol}}^{-1}\mathbf{Z}\mathbf{A}_{k_1}^{-1} & \cdots & \mathbf{A}_{k_{ol}}^{-1} - \mathbf{A}_{k_{ol}}^{-1}\mathbf{Z}\mathbf{A}_{k_{ol}}^{-1} \end{bmatrix}, \quad (5.C.21)$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{A}_{k_1}^{-1}\mathbf{Z} \\ \vdots \\ \mathbf{A}_{k_{ol}}^{-1}\mathbf{Z} \end{bmatrix}, \quad (5.C.22)$$

$$\mathbf{Y} = -\begin{bmatrix} \mathbf{Z}\mathbf{A}_{k_1}^{-1} & \cdots & \mathbf{Z}\mathbf{A}_{k_{ol}}^{-1} \end{bmatrix}, \quad (5.C.23)$$

$$\mathbf{Z} = \frac{1}{\widehat{\Delta}} \begin{bmatrix} n_2 + \sum_{k \in \mathcal{D}_{ol}} \frac{1}{\Delta_i} & \sum_{k \in \mathcal{D}_{ol}} \frac{F_{21}(i) + \epsilon}{\Delta_i} \\ \sum_{k \in \mathcal{D}_{ol}} \frac{F_{12}(i) + \epsilon}{\Delta_i} & n_1 + \sum_{k \in \mathcal{D}_{ol}} \frac{1}{\Delta_i} \end{bmatrix}, \tag{5.C.24}$$

where

$$\Delta_i \triangleq \det(\mathbf{A}_i) = 1 - \big(F_{21}(i) + \epsilon\big)\big(F_{12}(i) + \epsilon\big), \tag{5.C.25}$$

$$\widehat{\Delta} \triangleq \left(n_1 + \sum_{k \in \mathcal{D}_{ol}} \frac{1}{\Delta_i}\right)\left(n_2 + \sum_{k \in \mathcal{D}_{ol}} \frac{1}{\Delta_i}\right) - \left(\sum_{k \in \mathcal{D}_{ol}} \frac{F_{21}(i) + \epsilon}{\Delta_i}\right)\left(\sum_{k \in \mathcal{D}_{ol}} \frac{F_{12}(i) + \epsilon}{\Delta_i}\right), \tag{5.C.26}$$

$$\mathbf{A}_i^{-1} = \begin{bmatrix} \frac{1}{\Delta_i} & -\frac{F_{21}(i)+\epsilon}{\Delta_i} \\ -\frac{F_{12}(i)+\epsilon}{\Delta_i} & \frac{1}{\Delta_i} \end{bmatrix}. \tag{5.C.27}$$

Thus, from (5.C.11) and (5.C.15),

$$\mathbf{P} = \mathbf{X}\mathbf{p_t} = \begin{bmatrix} \mathbf{A}_{k_1}^{-1}\mathbf{Z}\mathbf{p_t} & \dots & \mathbf{A}_{k_{ol}}^{-1}\mathbf{Z}\mathbf{p_t} \end{bmatrix}^T, \tag{5.C.28}$$

and

$$\frac{\partial \mathbf{P}}{\partial \epsilon} = -\mathbf{W}\left(\frac{\partial \mathbf{A}}{\partial \epsilon}\right)\mathbf{X}, \tag{5.C.29}$$

$$= \begin{bmatrix} \sum_{i=k_1}^{k_{ol}} \mathbf{A}_{k_1}^{-1}\mathbf{Z}\mathbf{A}_i^{-1}\mathbf{G}\mathbf{A}_i^{-1}\mathbf{Z}\mathbf{p_t} - \mathbf{A}_{k_1}^{-1}\mathbf{G}\mathbf{A}_{k_1}^{-1}\mathbf{Z}\mathbf{p_t} \\ \vdots \\ \sum_{i=k_1}^{k_{ol}} \mathbf{A}_{k_{ol}}^{-1}\mathbf{Z}\mathbf{A}_i^{-1}\mathbf{G}\mathbf{A}_i^{-1}\mathbf{Z}\mathbf{p_t} - \mathbf{A}_{k_{ol}}^{-1}\mathbf{G}\mathbf{A}_{k_{ol}}^{-1}\mathbf{Z}\mathbf{p_t} \end{bmatrix}, \tag{5.C.30}$$

where

$$\mathbf{G} = \frac{\partial\, \mathbf{A}_i}{\partial\, \epsilon} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \forall\ i = 1, \ldots, N. \tag{5.C.31}$$

Therefore, for $k = k_1, \ldots, k_{ol}$,

$$\mathbf{p}(k) = \begin{bmatrix} p_1(k) \\ p_2(k) \end{bmatrix} = \mathbf{A}_k^{-1}\mathbf{Z}\mathbf{p_t} \tag{5.C.32}$$

and

$$\begin{aligned} \mathbf{p}'(k) &= \frac{\partial}{\partial \epsilon}\,\mathbf{p}(k), && (5.C.33) \\ &= \begin{bmatrix} p_1'(k) \\ p_2'(k) \end{bmatrix}, && (5.C.34) \\ &= \sum_{i=k_1}^{k_{ol}} \mathbf{A}_k^{-1}\mathbf{Z}\mathbf{A}_i^{-1}\mathbf{G}\mathbf{A}_i^{-1}\mathbf{Z}\mathbf{p_t} - \mathbf{A}_k^{-1}\mathbf{G}\mathbf{A}_k^{-1}\mathbf{Z}\mathbf{p_t}, && (5.C.35) \\ &= \mathbf{A}_k^{-1}\mathbf{Z}\sum_{i=k_1}^{k_{ol}} \mathbf{A}_i^{-1}\mathbf{G}\mathbf{A}_i^{-1}\mathbf{Z}\mathbf{p_t} - \mathbf{A}_k^{-1}\mathbf{G}\mathbf{A}_k^{-1}\mathbf{Z}\mathbf{p_t} && (5.C.36) \end{aligned}$$

Consider the extent of partitioning $J(k)$ at any frequency k. For frequencies where $J(k) = 0$, at least one of the users has zero power allocation and that will not change with change in uncertainty. Thus $\frac{\partial}{\partial \epsilon} J(k) = 0$ for $k \in \mathcal{D}_1 \cup \mathcal{D}_2$. Also, in cases when n_1 or n_2 change due to some frequency $\bar{k}$ dropping from the set $\mathcal{D}_{ol}$, $J(\bar{k})$ increases from some negative value to zero.

Now consider the extent of partitioning for frequencies where both users have non-zero power allocation. Differentiating the extent of partitioning for frequency $k \in \mathcal{D}_{ol}$ with respect to ϵ,

$$\begin{aligned}
\frac{\partial}{\partial \epsilon} J(k) &= \frac{\partial}{\partial \epsilon} \left(-p_1(k) p_2(k)\right), && (5.C.37)\\
&= -p_1'(k) p_2(k) + p_1(k) p_2'(k), && (5.C.38)\\
&= -\begin{bmatrix} p_1(k) \\ p_2(k) \end{bmatrix}^T \begin{bmatrix} p_2'(k) \\ p_1'(k) \end{bmatrix}, && (5.C.39)\\
&= -\begin{bmatrix} p_1(k) \\ p_2(k) \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1'(k) \\ p_2'(k) \end{bmatrix}, && (5.C.40)\\
&= -\mathbf{p}(k)^T \mathbf{G} \mathbf{p}'(k), && (5.C.41)\\
&= -\Big(\mathbf{A}_k^{-1} \mathbf{Z} \mathbf{p_t}\Big)^T \mathbf{G} \Big(- \mathbf{A}_k^{-1} \mathbf{G} \mathbf{A}_k^{-1} \mathbf{Z} \mathbf{p_t} \\
&\qquad + \mathbf{A}_k^{-1} \mathbf{Z} \sum_{i=k_1}^{k_{ol}} \mathbf{A}_i^{-1} \mathbf{G} \mathbf{A}_i^{-1} \mathbf{Z} \mathbf{p_t} \Big), && (5.C.42)\\
&= \mathbf{p_t}^T \mathbf{Z}^T \mathbf{A}_k^{-1^T} \mathbf{G} \mathbf{A}_k^{-1} \Big(- \mathbf{Z} \sum_{i=k_1}^{k_{ol}} \mathbf{A}_i^{-1} \mathbf{G} \mathbf{A}_i^{-1} \mathbf{G}^{-1} \mathbf{A}_k \\
&\qquad + \mathbf{I} \Big) \mathbf{G} \mathbf{A}_k^{-1} \mathbf{Z} \mathbf{p_t}. && (5.C.43)
\end{aligned}$$

Let $\mathbf{q}_k \triangleq \mathbf{G}\mathbf{A}_k^{-1}\mathbf{Z}\mathbf{p_t}$. Using $\mathbf{G}^T = \mathbf{G}^{-1} = \mathbf{G}$, $\mathbf{G}\mathbf{A}_i^{-1}\mathbf{G} = \mathbf{A}_i^{-T}$ and $\mathbf{G}\mathbf{A}_k\mathbf{G} = \mathbf{A}_k^T$, (5.C.43) can be written as

$$\frac{\partial}{\partial \epsilon} J(k) = \mathbf{q}_k^T \left(\mathbf{A}_k^{-1} - \mathbf{A}_k^{-1} \mathbf{Z} \textstyle\sum_{i=k_1}^{k_{ol}} \mathbf{A}_i^{-1} \mathbf{A}_i^{-T} \mathbf{A}_k^T \right) \mathbf{q}_k \qquad (5.C.44)$$

Let $\mathbf{M}_k = \sum_{i=k_1}^{k_{ol}} \mathbf{A}_i^{-1}\mathbf{A}_i^{-T}\mathbf{A}_k^T$ and $\mathbf{Q}_k = \mathbf{A}_k^{-1} - \mathbf{A}_k^{-1}\mathbf{Z}\mathbf{M}_k$. When $n_{ol} = o(N)$ (i.e., when $\lim_{N\to\infty} \frac{n_{ol}}{N} = 0$), the total number of frequencies, $n_1 + n_2 = \mathcal{O}(N)$. Since $\mathbf{A}_i^{-1}\mathbf{A}_i^{-T}\mathbf{A}_k^T = \mathcal{O}(1)$ for each i and k, $\mathbf{M}_k = \mathcal{O}(n_{ol})$ and

$\mathbf{Z} = \mathcal{O}(1/N)$. Thus,

$$\lim_{N\to\infty} \mathbf{A}_k^{-1}\mathbf{Z}\mathbf{M}_k = \mathbf{0} \tag{5.C.45}$$

which means that

$$\lim_{N\to\infty} \mathbf{Q}_k + \mathbf{Q}_k^T = \mathbf{A}_k^{-1} + \mathbf{A}_k^{-T} \succ \mathbf{0} \tag{5.C.46}$$

from the convergence condition. Thus, $\mathbf{x}^T\mathbf{Q}_k\mathbf{x} > 0 \ \forall \mathbf{x} \in \mathbb{R}^{2\times 1}$ as its symmetric part $\mathbf{Q}_k + \mathbf{Q}_k^T$ is positive definite [107]. Hence, $\frac{\partial}{\partial \epsilon} J(k) \geq 0$ when $N \to \infty$, with equality when $J(k) = 0$.

Derivation of Inverses

Consider

$$\mathbf{C}\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix}\mathbf{I}_2 & \dots & \mathbf{I}_2\end{bmatrix} \begin{bmatrix} \mathbf{A}_{k_1}^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_{k_{ol}}^{-1} \end{bmatrix} \begin{bmatrix} -\mathbf{I}_2 \\ \vdots \\ -\mathbf{I}_2 \end{bmatrix} \tag{5.C.47}$$

$$= \begin{bmatrix}\mathbf{A}_{k_1}^{-1} & \dots & \mathbf{A}_{k_{ol}}^{-1}\end{bmatrix} \begin{bmatrix} -\mathbf{I}_2 \\ \vdots \\ -\mathbf{I}_2 \end{bmatrix} = -\sum_{k\in\mathcal{D}_{ol}} \mathbf{A}_i^{-1} \tag{5.C.48}$$

$$= -\begin{bmatrix} \displaystyle\sum_{k\in\mathcal{D}_{ol}} \frac{1}{\Delta_i} & \displaystyle -\sum_{k\in\mathcal{D}_{ol}} \frac{F_{21}(i)+\epsilon}{\Delta_i} \\ \displaystyle -\sum_{k\in\mathcal{D}_{ol}} \frac{F_{12}(i)+\epsilon}{\Delta_i} & \displaystyle\sum_{k\in\mathcal{D}_{ol}} \frac{1}{\Delta_i} \end{bmatrix} \tag{5.C.49}$$

where Δ_i is defined in (5.C.25). Thus,

$$\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} n_1 + \sum_{k \in \mathcal{D}_{ol}} \frac{1}{\Delta_i} & -\sum_{k \in \mathcal{D}_{ol}} \frac{F_{21}(i) + \epsilon}{\Delta_i} \\ -\sum_{k \in \mathcal{D}_{ol}} \frac{F_{12}(i) + \epsilon}{\Delta_i} & n_2 + \sum_{k \in \mathcal{D}_{ol}} \frac{1}{\Delta_i} \end{bmatrix} \tag{5.C.50}$$

and

$$\begin{aligned} \mathbf{Z} &= \left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \\ &= \frac{1}{\widehat{\Delta}} \begin{bmatrix} n_2 + \sum_{k \in \mathcal{D}_{ol}} \frac{1}{\Delta_i} & \sum_{k \in \mathcal{D}_{ol}} \frac{F_{21}(i) + \epsilon}{\Delta_i} \\ \sum_{k \in \mathcal{D}_{ol}} \frac{F_{12}(i) + \epsilon}{\Delta_i} & n_1 + \sum_{k \in \mathcal{D}_{ol}} \frac{1}{\Delta_i} \end{bmatrix} \end{aligned} \tag{5.C.51}$$

where $\widehat{\Delta}$ is defined in (5.C.26). Therefore,

$$\mathbf{X} = -\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \tag{5.C.52}$$

$$= \begin{bmatrix} \mathbf{A}_{k_1}^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_{k_{ol}}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{Z} \\ \vdots \\ \mathbf{Z} \end{bmatrix} \tag{5.C.53}$$

$$= \begin{bmatrix} \mathbf{A}_{k_1}^{-1}\mathbf{Z} \\ \vdots \\ \mathbf{A}_{k_{ol}}^{-1}\mathbf{Z} \end{bmatrix} \tag{5.C.54}$$

and

$$\mathbf{Y} = -\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} \tag{5.C.55}$$

$$= -\begin{bmatrix}\mathbf{Z} & \dots & \mathbf{Z}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{k_1}^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_{k_{ol}}^{-1}\end{bmatrix} \tag{5.C.56}$$

$$= -\begin{bmatrix}\mathbf{Z}\mathbf{A}_{k_1}^{-1} & \cdots & \mathbf{Z}\mathbf{A}_{k_{ol}}^{-1}\end{bmatrix} \tag{5.C.57}$$

Also,

$$\mathbf{W} - \mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} \tag{5.C.58}$$

$$= \begin{bmatrix}\mathbf{A}_{k_1}^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_{k_{ol}}^{-1}\end{bmatrix}\begin{bmatrix}-\mathbf{I}_2 \\ \vdots \\ -\mathbf{I}_2\end{bmatrix}\begin{bmatrix}\mathbf{Z}\mathbf{A}_{k_1}^{-1} \\ \vdots \\ \mathbf{Z}\mathbf{A}_{k_{ol}}^{-1}\end{bmatrix}^T \tag{5.C.59}$$

$$= -\begin{bmatrix}\mathbf{A}_{k_1}^{-1} \\ \vdots \\ \mathbf{A}_{k_{ol}}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{Z}\mathbf{A}_{k_1}^{-1} & \cdots & \mathbf{Z}\mathbf{A}_{k_{ol}}^{-1}\end{bmatrix} \tag{5.C.60}$$

$$= -\begin{bmatrix}\mathbf{A}_{k_1}^{-1}\mathbf{Z}\mathbf{A}_{k_1}^{-1} & \cdots & \mathbf{A}_{k_1}^{-1}\mathbf{Z}\mathbf{A}_{k_{ol}}^{-1} \\ \vdots & & \vdots \\ \mathbf{A}_{k_{ol}}^{-1}\mathbf{Z}\mathbf{A}_{k_1}^{-1} & \cdots & \mathbf{A}_{k_{ol}}^{-1}\mathbf{Z}\mathbf{A}_{k_{ol}}^{-1}\end{bmatrix} \tag{5.C.61}$$

Hence,

$$\mathbf{W} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} \tag{5.C.62}$$

$$= \begin{bmatrix}\mathbf{A}_{k_1}^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_{k_{ol}}^{-1}\end{bmatrix} - \begin{bmatrix}\mathbf{A}_{k_1}^{-1}\mathbf{Z}\mathbf{A}_{k_1}^{-1} & \cdots & \mathbf{A}_{k_1}^{-1}\mathbf{Z}\mathbf{A}_{k_{ol}}^{-1} \\ \vdots & & \vdots \\ \mathbf{A}_{k_{ol}}^{-1}\mathbf{Z}\mathbf{A}_{k_1}^{-1} & \cdots & \mathbf{A}_{k_{ol}}^{-1}\mathbf{Z}\mathbf{A}_{k_{ol}}^{-1}\end{bmatrix} \tag{5.C.63}$$

$$= \begin{bmatrix} \mathbf{A}_{k_1}^{-1} - \mathbf{A}_{k_1}^{-1}\mathbf{Z}\mathbf{A}_{k_1}^{-1} & \cdots & -\mathbf{A}_{k_1}^{-1}\mathbf{Z}\mathbf{A}_{k_{ol}}^{-1} \\ \vdots & & \vdots \\ -\mathbf{A}_{k_{ol}}^{-1}\mathbf{Z}\mathbf{A}_{k_1}^{-1} & \cdots & \mathbf{A}_{k_{ol}}^{-1} - \mathbf{A}_{k_{ol}}^{-1}\mathbf{Z}\mathbf{A}_{k_{ol}}^{-1} \end{bmatrix} \quad (5.C.64)$$

□

Appendix 5.D Proof of Theorem 5.2

Using [23, Corollary 3.1], the sum of the rates of the two users in the frequency k is quasi-convex only if $F_{21}(k)F_{12}(k) > 1/4$. Let C be the minimum number of frequencies occupied by any user. When there are only two users and a large number of frequencies, $C \gg 1$. If the condition $F_{21}(k)F_{12}(k) > \frac{1}{4}(1 + \frac{1}{C-1})^2$ is satisfied for some $C \geq 2$ for all frequencies $k \in \{1, \dots, N\}$ (thus satisfying $F_{21}(k)F_{12}(k) > 1/4$), then the Pareto optimal solution is FDMA [23, Theorem 3.3]. This needs to be satisfied for the worst-case channel coefficients which leads to (5.2.3). Thus, the solution moving closer to FDMA will improve the sum-rate of the system. From Lemma 5.1, the robust equilibrium moves closer to FDMA as uncertainty increases and thus will result in an improvement in sum-rate.

The Pareto optimal solution under this condition (which is FDMA) is constant under varying uncertainty bounds as such an uncertainty in the interference coefficients $F_{12}(k)$ and $F_{21}(k)$ does not affect the FDMA solution where there is no interference. Thus, an increase in sum-rate will result in an decrease in price of anarchy. □

Chapter 6

ROBUST IWFA FOR MIMO SYSTEMS

In chapters 4 and 5, a robust rate-maximization game in SISO frequency-selective Gaussian interference channels under bounded channel uncertainty was presented and analyzed. In this chapter, a robust framework for rate-maximization games in multi-antenna systems (MIMO Gaussian interference channels) based on the robust game model is presented.

The chapter begins with a description of the system model under which the robust waterfilling solution is developed. This is followed by the formulation of the robust MIMO rate-maximization game under bounded channel uncertainty. This framework is shown to be a modified MIMO rate-maximization game (Section 3.5). The equilibrium for this game is then presented, along with an iterative waterfilling algorithm to compute it. Sufficient conditions for the uniqueness of the equilibrium and convergence of the algorithm are then presented. Finally, simulation results are presented to demonstrate the behaviour of the algorithm.

6.1 System model

Consider a MIMO Gaussian interference channel composed of Q MIMO links. The signal vector $\mathbf{y}_q \in \mathbb{C}^{n_{R_q}\times 1}$ measured at the receiver of user q is

$$\mathbf{y}_q = \widetilde{\mathbf{H}}_{qq}\mathbf{x}_q + \sum_{r\neq q}\mathbf{H}_{rq}\mathbf{x}_r + \mathbf{n}_q \tag{6.1.1}$$

where $\widetilde{\mathbf{H}}_{qq} \in \mathbb{C}^{n_{R_q}\times n_{T_q}}$ is the direct-channel matrix between source q and destination q, $\mathbf{H}_{rq} \in \mathbb{C}^{n_{R_q}\times n_{T_r}}$ is the cross-channel matrix between source r and destination q, $\mathbf{x}_q \in \mathbb{C}^{n_{T_q}\times 1}$ is the signal vector transmitted by source q and $\mathbf{n}_q \in \mathbb{C}^{n_{R_q}\times 1}$ is the receiver noise vector of user q, which is assumed to be a zero-mean complex Gaussian vector with an arbitrary (nonsingular) covariance matrix $\mathbf{R}_{n_q}$. The second term in the right hand side of (6.1.1) is the multi-user interference (MUI) observed at the destination q, which is treated as additive spatially coloured noise at the receiver of user q.

The system is assumed to be quasi-stationary for the duration of the transmission. Each receiver is assumed to be able to measure accurately the covariance matrix of the noise plus MUI generated by the other users. The direct-channel matrix $\widetilde{\mathbf{H}}_{qq}$ is assumed to be *square* and *nonsingular*. It is estimated by receiver of user q and is assumed to have a bounded uncertainty of unknown distribution. The uncertainty set $\mathcal{H}_q$ of the direct-channel matrix $\widetilde{\mathbf{H}}_{qq}$ is deterministically modelled as an ellipsoid centered around the nominal value $\mathbf{H}_{qq}$,

$$\mathcal{H}_q \triangleq \{\widetilde{\mathbf{H}}_{qq} \triangleq \mathbf{H}_{qq} + \mathbf{\Delta}_q : \|\mathbf{\Delta}_q\|_F \leq \epsilon_q\} \tag{6.1.2}$$

Based on this information, each destination q computes the optimal covariance matrix $\mathbf{Q}_q \triangleq \mathrm{E}\{\mathbf{x}_q\mathbf{x}^H\}$ for its own link and transmits it back to its transmitter over a low bit-rate error-free feedback channel. From this optimal covariance matrix, the beamformer weights of the transmitter can be computed as

$$\mathbf{x}_q = \sum_{i=1}^{n_{T_q}} \lambda_{q_i}\mathbf{v}_{q_i} \tag{6.1.3}$$

where λ_{q_i} is the i-th eigenvalue of $\mathbf{Q}_q$ and $\mathbf{v}_{q_i}$ is its associated eigenvector.

The nominal information rate of user q, $R_q(\mathbf{Q}_q, \mathbf{Q}_{-q})$, for this system can be written as [63]

$$R_q(\mathbf{Q}_q, \mathbf{Q}_{-q}) = \log\det(\mathbf{I} + \mathbf{H}_{qq}^H\mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\mathbf{Q}_q) \tag{6.1.4}$$

where

$$\mathbf{R}_{-q}(\mathbf{Q}_{-q}) \triangleq \mathbf{R}_{\mathbf{n}_q} + \sum_{r\neq q}\mathbf{H}_{rq}\mathbf{Q}_r\mathbf{H}_{rq}^H \tag{6.1.5}$$

is the interference plus noise covariance matrix observed by destination q, and $\mathbf{Q}_{-q} \triangleq \{\mathbf{Q}_r\}_{r\neq q}$ is the set of covariance matrices of all users except the q-th user. Each player q competes rationally against other users in order to maximize its own information rate $R_q(\mathbf{Q}_q, \mathbf{Q}_{-q})$ by designing the optimal covariance matrix $\mathbf{Q}_q^\star$, given the constraint

$$\mathrm{E}\left\{\|\mathbf{x}_q\|_2^2\right\} = \mathrm{Tr}(\mathbf{Q}_q) \leq P_q \tag{6.1.6}$$

where P_q is the maximum average power transmitted in units of energy per transmission for user q.

6.2 Robust rate-maximization game formulation

The robust game model (Section 2.5) suggests that when players have uncertainties in their payoff functions, formulating their best response to the worst-case payoff functions leads to a stable equilibrium. Motivated by this approach, a *protection function* (which is a lower bound on the payoff function) is formulated for each user, which is then maximized by each user.

Defining the matrices $\mathbf{M}_q$ and $\mathbf{E}_q$ as

$$\mathbf{M}_q \triangleq \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}, \tag{6.2.1}$$

$$\mathbf{E}_q \triangleq \mathbf{I} + \mathbf{H}_{qq}^{-1}\boldsymbol{\Delta}_q, \tag{6.2.2}$$

the protection function for user q, based on the channel uncertainty model in (6.1.2), is formulated as

$$\begin{aligned}
\widetilde{R}_q(\mathbf{Q}_q, \mathbf{Q}_{-q}) &= \log\det(\mathbf{I} + \widetilde{\mathbf{H}}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\widetilde{\mathbf{H}}_{qq}\mathbf{Q}_q), && (6.2.3)\\
&= \log\det(\mathbf{I} + \mathbf{E}_q^H \mathbf{M}_q \mathbf{E}_q \mathbf{Q}_q), && (6.2.4)\\
&= \sum_{i=1}^{n_q} \log \lambda_i(\mathbf{I} + \mathbf{E}_q^H \mathbf{M}_q \mathbf{E}_q \mathbf{Q}_q), && (6.2.5)\\
&= \sum_{i=1}^{n_q} \log \Big(1 + \lambda_i(\mathbf{E}_q^H \mathbf{E}_q \mathbf{M}_q \mathbf{Q}_q)\Big), && (6.2.6)\\
&\geq \sum_{i=1}^{n_q} \log \Big(1 + \lambda_{\min}(\mathbf{E}_q^H \mathbf{E}_q)\lambda_i(\mathbf{M}_q \mathbf{Q}_q)\Big). && (6.2.7)
\end{aligned}$$

where (6.2.4) follows from (6.2.1) and (6.2.2); (6.2.5) follows from [107, Theorem 1.2.12]; (6.2.6) follows from Weyl's Theorem [107, Theorem 4.3.1];[1] and (6.2.7) follows from [109, Fact 8.19.17].[2]

Now,

$$
\begin{aligned}
\lambda_{\min}(\mathbf{E}_q^H\mathbf{E}_q) &= \lambda_{\min}(\mathbf{I} + \mathbf{\Delta}_q^H\mathbf{H}_{qq}^{-H} + \mathbf{H}_{qq}^{-1}\mathbf{\Delta}_q \\
&\qquad + \mathbf{\Delta}_q^H\mathbf{H}_{qq}^{-H}\mathbf{H}_{qq}^{-1}\mathbf{\Delta}_q), && (6.2.8)\\
&\geq 1 + \lambda_{\min}(\mathbf{\Delta}_q^H\mathbf{H}_{qq}^{-H} + \mathbf{H}_{qq}^{-1}\mathbf{\Delta}_q) \\
&\qquad + \lambda_{\min}(\mathbf{\Delta}_q^H\mathbf{H}_{qq}^{-H}\mathbf{H}_{qq}^{-1}\mathbf{\Delta}_q), && (6.2.9)\\
&\geq 1 - 2\sigma_{\max}(\mathbf{H}_{qq}^{-1}\mathbf{\Delta}_q) \\
&\qquad + \lambda_{\min}(\mathbf{\Delta}_q^H\mathbf{H}_{qq}^{-H}\mathbf{H}_{qq}^{-1}\mathbf{\Delta}_q), && (6.2.10)\\
&\geq 1 - 2\sigma_{\max}(\mathbf{H}_{qq}^{-1}\mathbf{\Delta}_q) \\
&\qquad + \lambda_{\min}(\mathbf{H}_{qq}^{-H}\mathbf{H}_{qq}^{-1})\lambda_{\min}(\mathbf{\Delta}_q^H\mathbf{\Delta}_q), && (6.2.11)\\
&\geq 1 - 2\sigma_{\max}(\mathbf{H}_{qq}^{-1}\mathbf{\Delta}_q) && (6.2.12)\\
&\geq 1 - 2\sigma_{\min}(\mathbf{H}_{qq}^{-1})\sigma_{\max}(\mathbf{\Delta}_q), && (6.2.13)\\
&= 1 - 2\frac{\sigma_{\max}(\mathbf{\Delta}_q)}{\sigma_{\max}(\mathbf{H}_{qq})}, && (6.2.14)
\end{aligned}
$$

[1] Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{N\times N}$ be Hermitian. For each $k = 1, 2, \ldots, N$, we have

$$\lambda_k(\mathbf{A}) + \lambda_{\min}(\mathbf{B}) \leq \lambda_k(\mathbf{A}+\mathbf{B}) \leq \lambda_k(\mathbf{A}) + \lambda_{\max}(\mathbf{B}).$$

[2] Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{N\times N}$ be Hermitian and positive definite. For each $k = 1, 2, \ldots, N$, we have

$$\lambda_k(\mathbf{A})\lambda_{\min}(\mathbf{B}) \leq \lambda_k(\mathbf{AB}) \leq \lambda_k(\mathbf{A})\lambda_{\max}(\mathbf{B}).$$

$$\geq 1 - \frac{2\epsilon_q}{\sigma_{\max}(\mathbf{H}_{qq})}, \tag{6.2.15}$$

where (6.2.9) follows from Weyl's Theorem [107, 4.3.1]; (6.2.10) follows from [109, Fact 5.11.25]; (6.2.11) follows from [109, Fact 8.19.17]; (6.2.13) follows from [109, Proposition 9.6.6] and (6.2.15) follows from the definition of Frobenius norm.

Using (6.2.15) in (6.2.7), the protection function for user q can be formulated as

$$\widetilde{R}_q(\mathbf{Q}_q, \mathbf{Q}_{-q}) \geq \sum_{i=1}^{n_q} \log\Big(1 + \gamma_q \lambda_i(\mathbf{M}_q \mathbf{Q}_q)\Big), \tag{6.2.16}$$

$$= \log\det\Big(\mathbf{I} + \gamma_q \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\mathbf{Q}_q\Big), \tag{6.2.17}$$

where (6.2.17) follows from [107, Theorem 1.2.12] and γ_q is defined as

$$\gamma_q \triangleq 1 - \frac{2\epsilon_q}{\sigma_{\max}(\mathbf{H}_{qq})}. \tag{6.2.18}$$

Note that the lower bound indicated by γ_q could be too loose if the uncertainty bound ϵ_q is too high or if the largest singular value of the direct channel, $\sigma_{\max}(\mathbf{H}_{qq})$ is too small. In particular, this could lead to $\gamma_q \leq 0$. However, $\lambda_{\min}(\mathbf{E}_q^H \mathbf{E}_q) \geq 0$. Hence, the range of γ_q is limited to $0 < \gamma_q \leq 1$.

Based on the protection function in (6.2.17), the robust MIMO rate-maximization game can be mathematically formulated as

$$\mathscr{G}_{\text{rob}}^{\text{M}} \quad \begin{array}{ll} \max\limits_{\mathbf{Q}_q} & \log\det\left(\mathbf{I} + \gamma_q \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\mathbf{Q}_q\right) \\ \text{s.t.} & \mathbf{Q}_q \in \mathscr{Q}_q \end{array} \quad \forall q \in \Omega \tag{6.2.19}$$

where $\Omega \triangleq \{1, \ldots, Q\}$ is the set of the Q players (i.e. MIMO links), $R_q(\mathbf{Q}_q, \mathbf{Q}_{-q})$ is the payoff function of player q as given in (6.1.4) and the set of admissible strategies of player q, $\mathscr{Q}_q$, is defined as

$$\mathscr{Q}_q \triangleq \{\mathbf{Q} \in \mathbb{C}^{n_{T_q} \times n_{T_q}} : \mathbf{Q} \succeq \mathbf{0}, \quad \text{Tr}(\mathbf{Q}_q) = P_q\}. \tag{6.2.20}$$

The inequality constraint in (6.1.6) is replaced with the equality constraint in (6.2.20) as, at the optimum of each problem in (6.2.19), the constraint must be satisfied with equality [99].

Note that the quantity γ_q of user q is dependent only on its own direct-channel $\mathbf{H}_{qq}$ and its uncertainty bound ϵ_q, and thus does not need any additional information (other than the uncertainty bound), such as other users' transmit covariances or channel matrices, when computing the robust solutions. Furthermore, the quantity γ_q is related to the relative uncertainty in the direct channel matrices (determined by the ratio $\epsilon_q/\sigma_q(\mathbf{H}_{qq})$). In addition, this formulation has the advantage of not needing any additional computational hardware, as the eigendecomposition is performed anyway in every iteration of the algorithm when computing the waterfilling solutions. Moreover, the additional computational cost is not going to be significant, as the quantity γ_q needs to be computed only once, at the beginning of the game.

It can be observed that the robust game $\mathscr{G}_{\text{rob}}^{\text{M}}$ is equivalent to the nominal game described in Section 3.5 (and in [63]), with the modified channels $\{\gamma_q^{1/2}\mathbf{H}_{qq}\}_{q\in\Omega}$ instead of the original channels $\{\mathbf{H}_{qq}\}_{q\in\Omega}$.

6.3 Robust-optimization equilibrium

Recall from Section 3.5 that the solution to the nominal game is the Nash equilibrium. In this game, given $\mathbf{Q}_{-q} \in \mathscr{Q}_{-q} \triangleq \mathscr{Q}_1 \times \cdots \times \mathscr{Q}_{q-1} \times \mathscr{Q}_{q+1} \times \cdots \mathscr{Q}_Q$, the optimum action profile of the players $\{\mathbf{Q}_q^\star\}_{q\in\Omega}$ at equilibrium must satisfy, $\forall q \in \Omega$,

$$\mathbf{Q}_q^\star = \text{RWF}_{\text{q}}^{\text{M}}(\mathbf{Q}_1^\star, \ldots, \mathbf{Q}_{q-1}^\star, \mathbf{Q}_{q+1}^\star, \ldots, \mathbf{Q}_Q^\star) = \text{RWF}_{\text{q}}^{\text{M}}(\mathbf{Q}_{-q}^\star). \tag{6.3.1}$$

The robust waterfilling operator $\text{RWF}_{\text{q}}^{\text{M}}(\cdot)$ is defined as

$$\text{RWF}_{\text{q}}^{\text{M}}(\mathbf{Q}_{-q}) \triangleq \mathbf{U}_q(\mu_q \mathbf{I} - \tfrac{1}{\gamma_q}\mathbf{D}_q^{-1})^+ \mathbf{U}_q^H \tag{6.3.2}$$

where μ_q is chosen to satisfy $\text{Tr}\left((\mu_q \mathbf{I} - \frac{1}{\gamma_q}\mathbf{D}_q^{-1})^+\right) = P_q$. The unitary matrix of eigenvectors $\mathbf{U}_q = \mathbf{U}_q(\mathbf{Q_{-q}}) \in \mathbb{C}^{n_{T_q}\times n_{T_q}}$ and the diagonal matrix $\mathbf{D}_q = \mathbf{D}_q(\mathbf{Q}_{-q}) \in \mathbb{R}_{++}^{n_{T_q}\times n_{T_q}}$ are calculated from the eigendecomposition

$$\mathbf{U}_q \mathbf{D}_q \mathbf{U}_q^H \triangleq \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}. \tag{6.3.3}$$

Given the MIMO system in (6.1.1), the non-negative matrix $\mathbf{S}_\gamma \in \mathbb{R}_+^{Q\times Q}$ is defined as

$$[\mathbf{S}_\gamma]_{qr} \triangleq \begin{cases} \frac{1}{\gamma_q}\,\rho(\mathbf{H}_{rq}^H\mathbf{H}_{qq}^{-H}\mathbf{H}_{qq}^{-1}\mathbf{H}_{rq}), & \text{if } r \neq q, \\ 0, & \text{otherwise} \end{cases} \tag{6.3.4}$$

The sufficient condition for existence and uniqueness of the equilibrium of game $\mathscr{G}_{\text{rob}}^{\text{M}}$ is given by the following theorem:

THEOREM 6.1. *Game $\mathscr{G}^{\text{rob}}$ has at least one equilibrium for any feasible set of channel matrices and transmit powers of the users. Furthermore, the equilibrium is unique if*

$$\rho(\mathbf{S}_\gamma) < 1 \tag{6.3.5}$$

where $\mathbf{S}_\gamma$ is defined in (6.3.4)*.*

Proof. Refer [63, Theorem 6]. □

It can be verified that the above condition reduces to the nominal condition (3.5.18) when there is no uncertainty ($\gamma_q = 1\ \forall q \in \Omega$). An iterative algorithm to compute the equilibrium is presented and characterized in the following section.

6.3.1 Iterative algorithm for robust waterfilling

Let the discrete set $\mathcal{T} = \mathbb{N}_+ = 1, 2, \ldots$ be the set of times at which one or more users update their strategies. Let $\mathbf{Q}_q^{(n)}$ denote the set of covariance matrices of user q at the n-th iteration, and let $\mathcal{T}_q \subseteq \mathcal{T}$ denote the set of

Algorithm 6.1 – Robust MIMO Iterative Waterfilling Algorithm

Input:

Ω: Set of users in the system

$\mathscr{Q}_q$: Set of admissible strategies of user q

$\mathcal{T}_q$: Set of time instants n when the covariance matrix $\mathbf{Q}_q^{(n)}$ of user q is updated

T: Number of iterations for which the algorithm is run

$\tau_r^q(n)$: Time of the most recent covariance matrix of user r available to user q at time n

$\mathrm{RWF}_\mathrm{q}^\mathrm{M}(\cdot)$: Robust waterfilling operation in (6.3.2)

Initialization: $n = 0$ and $\mathbf{Q}_q^{(0)} \leftarrow$ any $\mathbf{Q} \in \mathscr{Q}_q, \qquad \forall q \in \Omega$

for $n = 0$ to T **do**

$$\mathbf{Q}_q^{(n+1)} = \begin{cases} \mathrm{RWF}_\mathrm{q}^\mathrm{M}\left(\mathbf{Q}_{-q}^{(\tau^q(n))}\right), & \text{if } n \in \mathcal{T}_q, \\ \mathbf{Q}_q^{(n)}, & \text{otherwise} \end{cases} \qquad \forall q \in \Omega,$$

end for

time instants n when the strategy $\mathbf{Q}_q^{(n)}$ of user q is updated. Let $\tau_r^q(n)$ denote the time when the most recently perceived interference from user r was computed by user q at time n (Note that $0 \leq \tau_r^q(n) \leq n$). Hence, if user q updates its strategy at time n, then

$$\mathbf{Q}_{-q}^{\left(\tau^q(n)\right)} \triangleq \left(\mathbf{Q}_1^{\left(\tau_1^q(n)\right)}, \ldots, \mathbf{Q}_{q-1}^{\left(\tau_{q-1}^q(n)\right)}, \mathbf{Q}_{q+1}^{\left(\tau_{q+1}^q(n)\right)}, \ldots, \mathbf{Q}_Q^{\left(\tau_Q^q(n)\right)}\right). \quad (6.3.6)$$

A fully distributed asynchronous iterative algorithm to compute the equilibrium of game $\mathscr{G}_{\mathrm{rob}}^{\mathrm{M}}$ is described in Algorithm 6.1. The convergence of Algorithm 6.1 is guaranteed under the following sufficiency condition:

THEOREM 6.2. *The robust MIMO iterative waterfilling algorithm, described in Algorithm 6.1 converges to the unique equilibrium of game* $\mathscr{G}_{\mathrm{rob}}^{\mathrm{M}}$ *as* $T \to \infty$

for any set of feasible initial conditions if (6.3.5) *is satisfied.*

Proof. Refer [63, Theorem 7]. □

When the relative uncertainties, i.e, the ratio $\epsilon_q/\sigma_q(\mathbf{H}_{qq})$, of all users is the same, the quantities γ_q of all users is identical. In this case, the sufficient condition in (6.3.5) can be simplified as follows:

Corollary 6.2.1. *When the uncertainties of all the users is identical, i.e., when $\gamma_q = \gamma$, $\forall q \in \Omega$, the sufficient condition for the uniqueness of the equilibrium and the guaranteed convergence of Algorithm 6.1, described in* (6.3.5)*, reduces to*

$$\rho(\mathbf{S}) < \gamma \tag{6.3.7}$$

where $\mathbf{S}$ *is defined as*

$$[\mathbf{S}]_{qr} \triangleq \begin{cases} \rho(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{-H} \mathbf{H}_{qq}^{-1} \mathbf{H}_{rq}), & \text{if } r \neq q, \\ 0, & \text{otherwise} \end{cases} \tag{6.3.8}$$

This result helps analyze the effect of uncertainty on the set of channel matrices for which the equilibrium is guaranteed to be unique and Algorithm 6.1 is guaranteed to converge. In the absence of uncertainty, this occurs when $\rho(\mathbf{S}) < 1$, (Corollary 3.7.1). When the uncertainty bound of the system increases, the value of γ reduces, and thus, the set of matrices that satisfy (6.3.7) shrinks. Thus, to achieve a robust solution, there is a trade-off between allowed uncertainty and guaranteed convergence of the algorithm.

6.4 Simulation results

In this section, the average behaviour of the robust MIMO algorithm under different scenarios is investigated. The effect of the amount of uncertainty, number of users, the number of transmit/receive antennas of each user and the level of interference on the average sum-rate of the system are presented here. Also, these results are compared with the nominal solution (i.e. using the MIMO waterfilling algorithm (Algorithm 3.2) with erroneous channel matrices).

The simulation results are provided for a system with Q users averaged over 10000 trials with random channel matrices. The cross-channel matrices $\mathbf{H}_{rq} \in \mathbb{C}^{N_t \times N_r}$ are generated with elements drawn from $N_C(0,1)$ for $r \neq q$ and the direct-channel matrices $\mathbf{H}_{qq} \in \mathbb{C}^{N_t \times N_r}$ are generated with elements drawn from $N_C(0, d_r^2)$. The channel uncertainty model is $\widetilde{\mathbf{H}}_{qq} \triangleq \mathbf{H}_{qq} + \mathbf{\Delta}_q$ where $\|\mathbf{\Delta}_q\|_F \leq \epsilon$ (from (6.1.2)). The specific parameters are provided with each figure. It is to be noted that the quantity d_r is the ratio between the standard deviation of the elements of the random direct-channel matrices and the standard deviation of the elements of the random cross-channel matrices. A higher value of d_r indicates weaker interference in the system.

In Figure 6.1, it can be observed that the sum-rate under the robust solution improves with rise in uncertainty while the sum-rate under the nominal solution falls with increase in uncertainty. This gap in performance can be observed to be zero under zero uncertainty (since the two solutions coincide) and rise to about 1.2 nats/transmission when the uncertainty bound is 0.5.

The average number of iterations required to converge to the robust solution against the uncertainty bound of the system is depicted in Figure 6.2. It can be observed that the robust solution takes longer to converge with higher uncertainty in the system, rising from about 6 iterations at zero uncertainty to about 8 iterations when the uncertainty bound is 0.5.

In Figure 6.3, the average sum-rate of a system with 2 users is plotted against the number of transmit/receive antennas of each user. It can be observed that the average sum-rate of the robust solution increases with the number of antennas, from about 10 nats/transmission when there are 2 antennas to about 13 nats/transmission when it is increased to 6 antennas, as expected in MIMO systems. Furthermore, the robust waterfilling solution consistently performs better than the nominal solution for the observed number of transmit/receive antennas, retaining an improvement of about 1 nat/transmission.

Figure 6.4 demonstrates the effect of number of users on the average sum-rate of the system. Increasing the number of users from 2 users to 6 users results in a lower sum-rate, reducing from about 8.5 nats/transmission to about 6 nats/transmission. This is because a higher number of users in the system results in more interference for all users, given a fixed value of d_r. In addition, it can be observed that the robust solution performs better than the nominal solution regardless of the number of users in the system.

In Figure 6.5, the effect of the level of interference on the average sum-rate of the system is demonstrated. The average sum-rate at the ro-

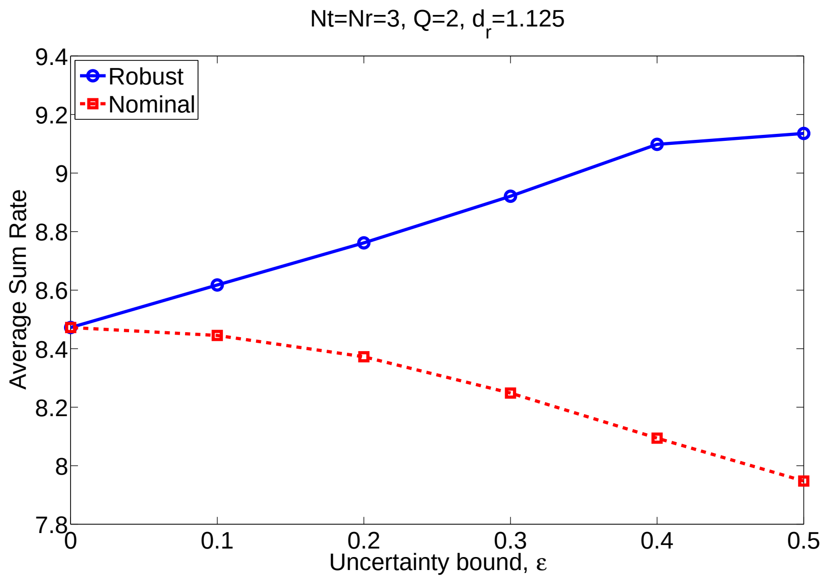


Figure 6.1: Sum-rate vs. channel uncertainty bound, ϵ.

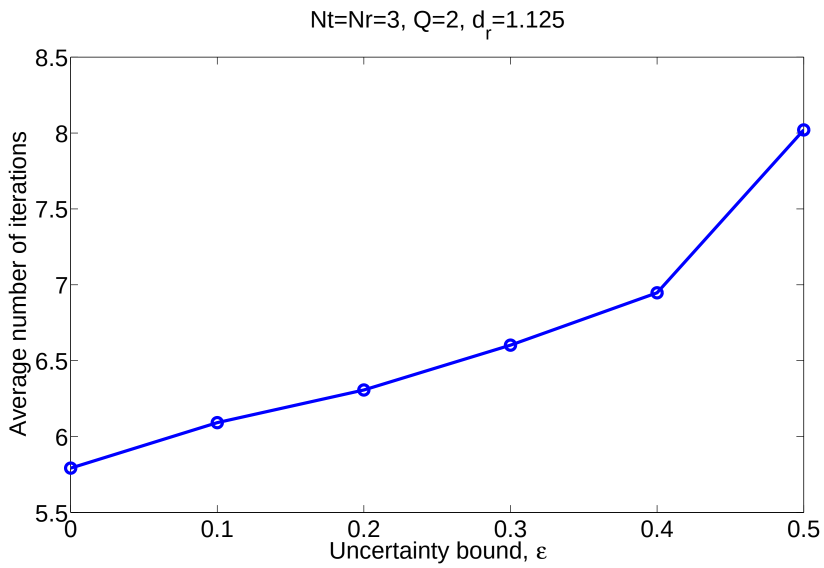


Figure 6.2: Average number of iterations for convergence vs. channel uncertainty bound.

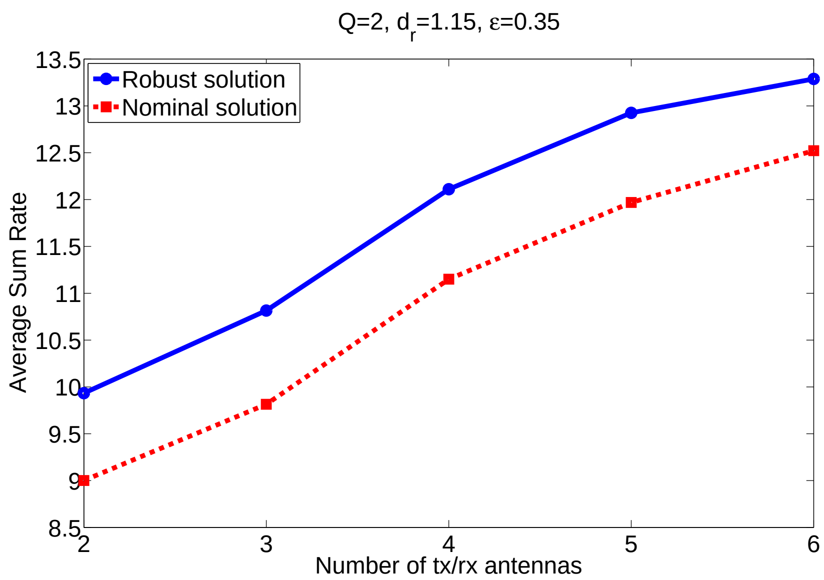


Figure 6.3: Sum-rate vs. number of transmit/receive antennas of each user.

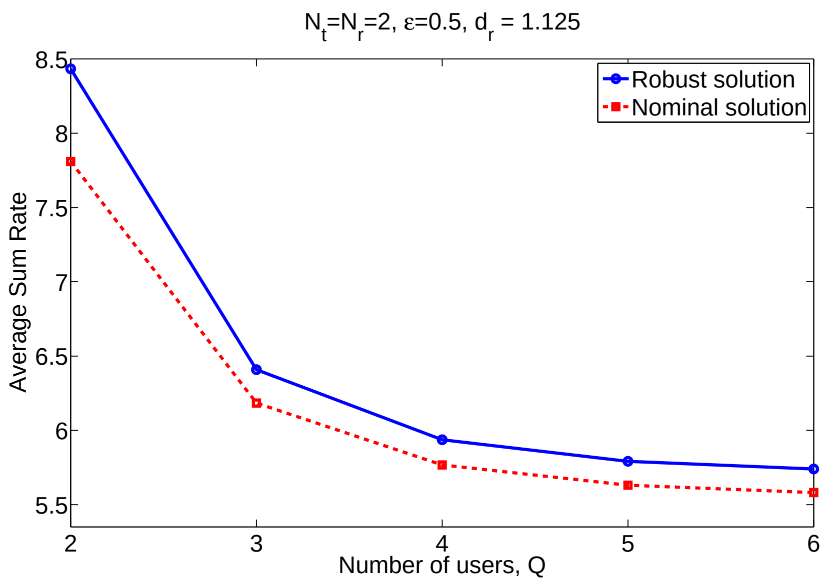


Figure 6.4: Sum-rate vs. number of users.

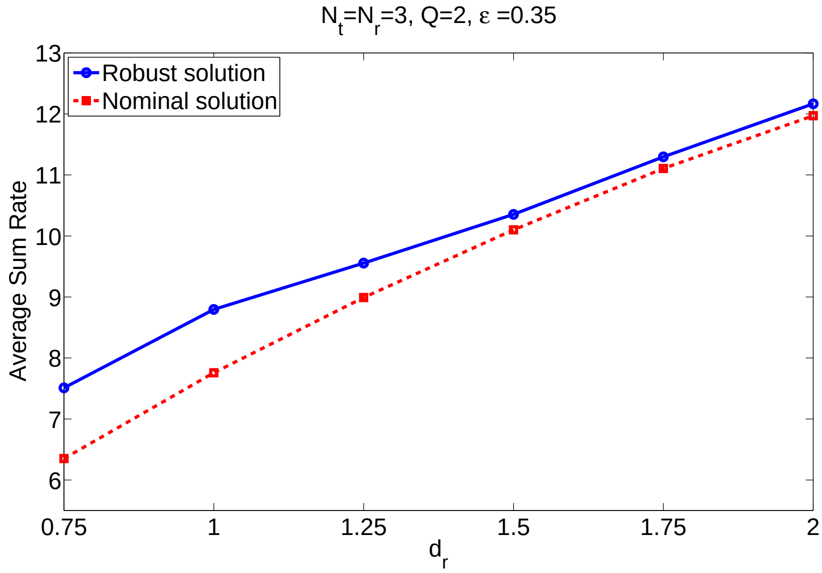


Figure 6.5: Sum-rate vs. direct-channel matrix standard deviation, d_r.

bust solution increases with reduction in interference, rising from about 7.5 nats/transmission when d_r is 0.75 to about 12 nats/transmission when d_r is 2. Note that a higher value of d_r indicates weaker interference in the system. It can also be observed that the gap in performance between the robust solution and the nominal solution is higher when the system has higher interference (1 nat/transmission when d_r is 0.75) and falls with reduction in interference (0.2 nats/transmission when d_r is 2). This can be explained as follows: the robust solution encourages each user to be less greedy, which results in lower interference for all users. In systems with stronger cross-channel matrices, this plays a greater role in determining the observed information rates of the users, when compared to systems with weak cross-channel matrices. Thus, when d_r increases, the robust solution moves closer to the nominal solution.

6.5 Summary

In this chapter, a robust formulation for the rate-maximization game in MIMO Gaussian interference channels in the presence of bounded channel uncertainty was developed. Inspired from the robust game model, best response of each user was based on a lower bound of the payoff function (protection function) and resulted in a distribution-free equilibrium solution. Furthermore, the robust MIMO rate-maximization game was observed to be equivalent to the nominal MIMO rate-maximization game with modified direct-channel matrices. This enabled the characterization and computation of the equilibrium utilizing an iterative waterfilling algorithm. Finally, simulation results demonstrated that the robust solution leads to better global performance, with higher achieved sum-rates.

Chapter 7

SUMMARY, CONCLUSIONS AND FUTURE WORK

In this chapter, the novel results of this thesis and the conclusions that can be drawn from them are summarized, followed by a discussion on future work that this work could lead to.

7.1 Summary and conclusions

The focus of this thesis has been the design of distributed algorithms to maximize the information rates of users in single-antenna and multi-antenna Gaussian interference channels in the presence of uncertainty in channel state information based on game theory.

In Chapter 1, the challenge of multiuser interference in next-generation wireless technologies was given as the motivation for the work contained in this thesis. Game theory has evolved as a suitable framework to design resource allocation schemes for such users. The majority of the current literature on game-theoretic solutions for resource allocation in wireless communications has assumed the availability of perfect channel knowledge, which is not possible in a practical situation. Hence, this thesis addressed the need for the analysis of the effect of imperfect channel knowledge on the performance of such game-theoretic methods and for the design of robust game-theoretic solutions which perform satisfactorily despite having imperfect channel knowledge.

In Chapter 2, relevant concepts from game theory have been briefly described. This included an overview of game theory and its underlying assumptions, followed by an introduction to the strategic noncooperative game and the concept of the Nash equilibrium. This was followed by a discussion on the idea of equilibrium efficiency and a few popular measures to quantify it. Finally, the limitations of the concept of the Nash equilibrium have been considered, and a robust optimization based approach to mitigating uncertainty in game theory called the robust game model has been introduced as the basis for the solutions presented in this thesis.

In Chapter 3, the conceptual foundations from fixed point theory, contraction mapping and information theory underpinning the work presented in this thesis have been summarized. In addition, this chapter introduced the specific game-theoretic problem formulations under which the issue of channel uncertainty is considered in this thesis. Finally, the effect of channel uncertainty on the performance of the MIMO iterative waterfilling algorithm, which demonstrates the need for robust solutions, has been investigated in this chapter.

In Chapter 4, a review of the current literature addressing the issue of uncertainty in rate-maximization games for interference channels has been presented. Following this, a robust formulation for the rate-maximization game in SISO frequency-selective Gaussian interference channels under bounded channel uncertainty has been developed. A distribution-free *robust optimization equilibrium* for this problem has been derived and proved to exist for all feasible channel realizations and to be unique under certain sufficient conditions. An iterative algorithm to compute the equilibrium in a distributed fashion has also been developed and shown to asymptotically converge when the equilibrium is unique. Simulation results have confirmed the behaviour of the algorithm and also have revealed an interesting effect of improvement in sum-rate of the system when channel uncertainty increases.

In Chapter 5, the improvement in sum-rate with increase in uncertainty that was observed in the previous chapter has been analytically investigated in a two-user setting. Based on the analysis of a simple two-frequency system, sufficient conditions for the improvement of sum-rate and price of anarchy in a system with asymptotically large number of frequencies with increase in uncertainty have been derived. In a nutshell, these results indicate that the robust-optimization equilibrium moves towards a frequency division multiple access (FDMA) solution as the uncertainty increases, thereby resulting in the improvement of sum-rate and price of anarchy when FDMA solutions are known to be globally optimal.

In Chapter 6, a robust formulation for the rate-maximization game in MIMO Gaussian interference channels in the presence of bounded channel uncertainty has been developed. The robust game thus developed has been shown to be equivalent to the nominal MIMO rate-maximization game with modified channel matrices. The robust-optimization equilibrium for this game and an iterative algorithm to compute it distributively have been presented and characterized. Numerical simulations on the behaviour of this solution have indicated that the robust solution (in the presence of channel uncertainty) performs better than the nominal solution (with perfect channel knowledge), similar to the robust SISO iterative waterfilling algorithm.

Based on the results presented in this thesis, it can be concluded that a robust game theoretic approach which unifies robust optimization techniques and traditional noncooperative game theory is a suitable approach to addressing channel uncertainty in rate-maximization games. Worst-case robust optimization is often too conservative in traditional single-objective optimization problems (such as beamformer design [110]) in order to ensure zero outage which results in a loss in performance. However, such a worst-case approach results in having the opposite effect in the multi-user game-theoretic setting where there are multiple coupled optimization problems.

The conservative solutions forced upon each user by worst-case optimization reduces their greediness which causes lesser interference to the other users in the system. When the system has significant interference (i.e. the cross-channels are comparable to the direct-channels), such a conservative approach by all users results in reduced interference for all users, which in turn leads to improved information rates for the users. This provides valuable insight into the design of better utility functions and mechanisms which, in some fashion, encourages reduced competition among selfish users (in the form of interference) and yields solutions which are closer to Pareto optimality, and yet enable distributed computation.

7.2 Future work

There are several directions in which the research presented in this thesis can be extended. The solutions presented here are for systems with open spectrum access, but they could be extended to the cognitive radio scenario, where there is an interference constraint which limits the interference observed at a licensed user. In addition, the robust MIMO iterative waterfilling algorithm presented in Chapter 6 is limited to square nonsingular channels, and can be extended to apply to systems with arbitrary channels. Another problem that could be considered is the robust rate-maximization game for MIMO systems with not just a total power constraint, but also a per-antenna power constraint. Also, the robust MIMO iterative waterfilling algorithm considers only uncertainty in the channel matrices, but not in the estimation of the covariance matrix of noise-plus-multiuser interference. Accounting for uncertainty in the estimation of the covariance matrix is particularly challenging, as such an estimation occurs in every iteration of the algorithm, and will lead to the breakdown of the definition of fixed points in such cases.

The robust solutions and techniques proposed in this thesis could also be extended to other power-control problems such as utilized power minimization subject to quality-of-service (QoS) constraints. This leads to problem formulations beyond the Nash equilibrium, in the area of nonlinear complementary problems and variational inequalities.

Another issue of interest is the behaviour of robust waterfilling algorithms in the multiple equilibria regime (when the cross-channels are very strong), which has received attention only recently for the situation when there is perfect channel knowledge [96]. Some of the questions that are of interest are: Is the multiplicity of equilibria affected by channel uncertainty? Are certain equilibria favoured at certain levels of uncertainty? Does the update order of the algorithm affect convergence and/or the equilibrium achieved? Does the initialization affect the convergence and/or equilibrium achieved?

Security considerations in the robust SISO iterative waterfilling algorithm presented in Chapter 4 are also of interest. This algorithm assumes public knowledge of the power allocation vectors, and has no safeguards against malicious users reporting false values. Designing mechanisms which discourage collusion and jamming though such means are necessary.

The methods proposed in this thesis also assume quasi-stationarity of the environment for the duration of the game. Extending these solutions to the dynamic case where channels could be changing states is of interest.

Another open problem of this general area is the scalability issue, as the Nash equilibrium has significant limitations when there are large number of users in the systems, each having large action-spaces.

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