Inverse Scattering Procedures for the
Reconstruction of One-Dimensional
Permittivity Range Profiles

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ABSTRACT

Inverse scattering is relevant to a very large class of problems, where the unknown structure of a scattering object is estimated by measuring the scattered field produced by known probing waves. Therefore, for more than three decades, the promises of non-invasive imaging inspection by electromagnetic probing radiations have been justifying a research interest on these techniques. Several application areas are involved, such as civil and industrial engineering, non-destructive testing and medical imaging as well as subsurface inspection for oil exploration or unexploded devices. In spite of this relevance, most scattering tomography techniques are not reliable enough to solve practical problems. Indeed, the nonlinear relationship between the scattered field and the object function and the robustness of the inversion algorithms are still open issues.

In particular, microwave tomography presents a number of specific difficulties that make it much more involved to implement than, for instance, X-ray tomography but, at the same time, offers unique advantages over other probing radiations, since good sensitivities are shown in the microwave region to important parameters in many diagnostic applications.

Our specific purpose, in the framework of a project financed by the Italian CNR (National Research Council) and the Italian Ministry of
University and Research, is to reconstruct the one-dimensional permittivity range profiles of architectural objects from microwave backscattering data on a specific frequency range. A very important task is to identify discontinuities in the reconstructed profile, since they carry essential information on possible heterogeneous inclusions in the building materials. The structure of this work is described next.

First, some useful background concepts regarding the inverse scattering problem are recalled. Then, the complete iterative procedure that we have developed for data inversion is described, which is based on a fully nonlinear data model in conjunction with an optimization technique. An experimental section, based on both numerical simulations and real measurements, is included to adequately validate the electromagnetic code as well as to assess the accuracy and efficiency of the reconstruction procedure. Finally, a detailed description of the in progress improvements to the present method along with future developments are presented.
1.1 THE INVERSE SCATTERING PROBLEM

The aim of collecting data is to gain meaningful information about physical systems or phenomena of interest. However, in many situations the quantities that we would like to figure out are different from those we are able to measure. If the measured data depends, in some way, on the quantities we want to retrieve, then the collected data should contain some information useful to recover those quantities. Starting from the data, the problem of trying to reconstruct the unknown entities is defined as an inverse problem.

Inverse scattering comprises a wide spectrum of applications where the unknown structure of a scattering object or some of its properties are estimated by using the scattered field excited by a known set of sources. For example, in the Computer Axial Tomography (CAT) we want to obtain transverse slices of a patient’s body and visualize them as images. In this case, the X-rays used to illuminate the body are partially transmitted and the opacity of the internal structures varies, resulting in a picture where the variation of the absorption coefficient in
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The body is depicted. The final aim is to reconstruct from the data the absorption as a function of position in the body. In the industrial field, non invasive testing by using microwaves is useful to check the quality of products such as aircraft skins or underground pipelines which need regular nondestructive tests to detect damages during their service lives that may be difficult or expensive to determine by other methods. In civil engineering, the integrity check of concrete structures, which may be weakened if the inner reinforcing steel is corroded, is of utmost importance as well as the damage level of wire ropes in suspension bridges that are subject to weather, vibration, and high loads. In geophysics, the inverse problem has an important role since the interior of the Earth is not directly observable but the surface appearance of waves that have been propagated through its interior is measurable. As a result, the measurements of seismic waves helps to determine the location of an earthquake’s epicentre or an oil deposit. It is therefore apparent that the scope of inverse problem theory is widely applicable in many diverse fields. This justifies the great scientific interest exhibited in the past years [1]-[5] and the fast growing research in the field toward newer and better techniques [6][7]-[10].

Among all of the mentioned facets of inverse scattering problems we want to focus our attention on microwave diffraction tomography. In fact, even though microwave tomography exhibits inherent difficulties both analytical and numerical [11]-[13], nonetheless it provides good sensitivity to important parameters as shown in several diagnostic applications [6],[14],[15]. This is the case, for example, of biomedical inquiring where it has been proved that the dielectric properties of a biological tissue are a strong indicator of its functional and pathological condition. The attractiveness of using microwaves instead of, for instance,
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X-rays, lies on their ability to better resolve the dielectric properties of tissues and therefore to give a crisper image based on differences. Moreover, this technique grants a safer alternative to ionizing radiations.

To better understand the nature of the differences between X-ray and microwave tomography, we need to look at the way the energy propagates inside the object. Let us consider the basic tomographic setup (Figure 1), where a narrow slice of the object is illuminated by an electromagnetic radiation and the intensity of the radiation traversing the object is measured. The data acquired from a single illumination at a given angle is referred to as projection. The tomographic reconstruction consists in computing the image of the slice from the collected projections. The word "tomography" itself is derived from the Greek tomos (slice) and graphia (to write).

In ideal situations, projections are a set of measurements of the integrated values of some parameter of the object, where integrations are evaluated along straight lines through the object resulting in the so called line integrals. This is the case of X-rays, where the radiation illuminating the
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object has the nature of straight rays (see Figure 2a). Typically the wavelength of the radiation is much smaller than the minimal feature size of the object under investigation (Figure 2b) and in the range spanning from 10 to 0.01 nanometers. The propagation of the energy is mostly affected by the absorption characteristic of the photoelectric effect, in which an x-ray photon releases all its energy to a tightly bound inner electron of an atom of the material.

On the other hand, the mechanism of interaction between microwaves and objects is intrinsically different (Figure 3a). Since the wavelength is comparable to the minimal feature size of the illuminated body (Figure 3b), the straight ray assumption suffers from the effects of refraction and diffraction and it is no more a proper model to describe the interaction. Energy propagation is characterized not only by refraction but also influenced by multipath effects. Each projection does not represent an integral along straight lines anymore, because when the geometrical laws

Figure 2 - a) X-ray tomography: (1) the impinging radiation, (2) straight-ray projection; b) Qualitative comparison between probing wavelength and object feature.
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of propagation don’t apply we cannot even use the concept of line integrals.

Figure 3 - a) Microwave tomography: (1) incident wave, (2) diffraction projection; b) the wavelength is comparable to the size of the investigated object .

Figure 4 - An object characterized by the function $o(x,y)$ is illuminated by using a ray defined by the angle $\theta$ and distance $d$ from the origin and it generates the projection $P_\theta(d)$.

Even if we have clearly stated the differences between X-rays and microwaves, it is worthy to briefly describe the basis of straight-ray
tomography, which helps us to introduce concepts and methodologies useful for diffraction tomography.

Let us therefore take into consideration how the radiation interacts with the object and produces the measured data (the projection). In other words, we want to address the *forward problem* for the system reported in Figure 4. In this simple proposed case, the object under investigation is modeled as a two-dimensional distribution \( o(x,y) \) of the attenuation constant for the X-ray radiation and the line integral gives the total attenuation suffered by a beam which travels through the object along a straight line characterized by the \((t, \theta)\) parameters [16]. The generic equation for the ray which probes the object under an angle equal to \( \theta \) is:

\[
t = x \cos \theta + y \sin \theta . \tag{1.1}
\]

where \( t \) is the perpendicular distance of the line from the origin. The resulting projection \( P_\theta(t) \), measured along a line orthogonal to the ray, is defined by the line integral (the *ray integral*)

\[
P_\theta(t) = \int_{Ray(t,\theta)} o(x,y) \, ds . \tag{1.2}
\]

Using the Dirac delta function \( \delta(.) \) it can be rewritten as

\[
P_\theta(t) = \int \int_{-\infty}^{\infty} \int \int_{-\infty}^{\infty} o(x,y) \, \delta(x \cos \theta + y \sin \theta - t) \, dx \, dy \tag{1.3}
\]
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which is obviously a function of $t$ for a given value of $\theta$ that defines the parallel projection of $o(x,y)$ for the angle $\theta$. The two-dimensional function $P_\theta(t)$ is called the Radon transform of $o(x,y)$; it is formed by combining a set of line integrals collected illuminating the object with parallel rays (see Figure 5).

![Figure 5](image)

**Figure 5** - The parallel projection $P_\theta(t)$ is measured by moving an X-ray source and detector along parallel lines on opposite sides of an object for a constant $\theta$.

At this point, we have completely described the forward problem but it is necessary to find out how to reconstruct the function $o(x,y)$ from its projections, therefore solving the inverse problem. The aim is to find the function $o(x,y)$ from its Radon transform. The fundamental theoretical basis is the Fourier Slice Theorem, which states that the one-dimensional Fourier transform of a parallel projection is equal to a slice of the two-dimensional Fourier transform of the investigated object. As a result, given the projection data, the unknown function characterizing the object can be estimated by just performing the two-dimensional inverse Fourier transform. We define the two-dimensional Fourier transform of the object function $o(x,y)$ as
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\[
O(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} o(x,y) e^{-j2\pi(ux+vy)} dx \, dy ,
\]

(1.4)

and the Fourier transform of the generic projection \( P_\theta(t) \) by

\[
S_\theta(w) = \int_{-\infty}^{\infty} P_\theta(t) e^{-j2\pi wt} dt .
\]

(1.5)

Setting for the projection \( P_\theta(t) \) the coordinate system \((t,s)\) as shown in Fig.6, we recall that eq. (1.2) can be written as

\[
P_\theta(t) = \int_{-\infty}^{\infty} o(t,s) ds ,
\]

(1.6)

under the transformation expressed by

\[
\left( \begin{array}{c} t \\ s \end{array} \right) = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) .
\]

(1.7)

Finally, it results

\[
S_\theta(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} o(t,s) e^{-j2\pi wt} dt ds = \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} o(x,y) e^{-j2\pi w(x \cos \theta + y \sin \theta)} dx dy = O(w,\theta)
\]

(1.8)
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where the right-hand side represents the two-dimensional Fourier transform at a spatial frequency \((u = w\cos \theta, v = w\sin \theta)\).

Figure 6 - The Fourier Slice Theorem relates the Fourier transform of the projection of an object to the Fourier transform of it along a radial line.

The result expressed by eq. (1.8) suggests that by calculating the Fourier transforms of the projections of an object at different angles, we can determine the values of \(O(u,v)\) along radial lines (Figure 6). Accordingly, if we were able to evaluate infinite projections, then \(O(u,v)\) would be completely known in the \(uv\)-plane. Once we have gained information about \(O(u,v)\), we can perform the inverse Fourier transform and hence we are able to recover \(o(x,y)\) as
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\[ o(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} O(u, v) e^{j2\pi(ux + vy)} du \, dv. \tag{1.9} \]

The Fourier Slice Theorem provides a clear and simple model of tomography and sets the basis for a practical implementation for reconstruction, which is known as the filtered backprojection algorithm [16].

However, as previously mentioned, when the object inhomogeneities become comparable to the wavelength of the probing radiation, it is no more possible to model the propagation as occurring along lines or rays, since energy transmission has to be treated in terms of wavefronts and fields scattered by the inhomogeneities. Moreover, the multipath effect, caused by the propagation through different parts of the refracting object, radically changes the scenario with respect to the X-ray case. Therefore, we need to reconsider the forward problem in this context.

Let us start analyzing the two dimensional case since, as we previously pointed out, tomography images are made of single slices of the object under investigation. This is also applicable in the case of a three-dimensional object which can be assumed to gently vary along one axis. The general form of a wave equation for an inhomogeneous medium is:

\[ \left[ \nabla^2 + k^2(\mathbf{r}) \right] u(\mathbf{r}) = 0, \tag{1.10} \]

where \( k(\mathbf{r}) \) is the two-dimensional propagation vector and \( u(\mathbf{r}) \) represents a two-dimensional plane wave of spatial frequency equal to \(|k(\mathbf{r})|\). If we
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ignore the effects of polarization, $k(r)$ can be considered as a scalar function representing the refractive index of the medium. Hence:

$$k(r) = k_0 n(r) = k_0 \left[1 + n_\delta(r)\right]$$ (1.11)

where $k_0$ represents the average wavenumber of the medium, $n(r)$ is the electromagnetic refractive index of the medium, and $n_\delta(r)$ denotes the refractive index deviation, which is zero outside the object. Eq. (1.10) now becomes:

$$\left(\nabla^2 + k_0^2\right)u(r) = -k_0^2 \left[n^2(r) - 1\right]u(r), \quad (1.12)$$

which can be rewritten as

$$\left(\nabla^2 + k_0^2\right)u(r) = -o(r)u(r). \quad (1.13)$$

Eq. (1.13) gives evidence to the unknown object function $o(r)$ describing the object under inspection. We can assume the field $u(r)$ to be composed by two different contributions, $u_0(r)$ and $u_s(r)$. The former is the incident field, which is a solution of the following equation

$$\left(\nabla^2 + k_0^2\right)u_0(r) = 0,$$ (1.14)

representing the field in the absence of any inhomogeneity. The latter component denotes the scattered field, which provides the contribution to
the total field exclusively due to the inhomogeneities. Reconsidering eq. (1.13) for the only scattering component, we observe that:

\[
\left( \nabla^2 + k_0^2 \right) u_s (r) = -u(r) o(r). \tag{1.15}
\]

Moreover, remembering that a solution of eq.(1.15) can be written by resorting to the following form

\[
\left( \nabla^2 + k_0^2 \right) g(r|r') = -\delta (r-r'), \tag{1.16}
\]

where \( g(r|r') \) is the Green’s function, we finally obtain

\[
u_s (r) = \int g(r|r') o(r') u(r') d r'. \tag{1.17}\]

Eq. (1.17) expresses the scattered field in terms of the total field and clearly states the challenging problem of recovering \( o(r) \), which can not be often solved exactly. This aspect will be analyzed in detail in the following section.
1.2 LINEARIZING THE PROBLEM

The inhomogeneous integral equation we have previously found represents the scattered field \( u_s(r) \) as a function of the unknown object described by \( \rho(r) \); it cannot be solved directly since the term \( u(r) \) contains the term \( u_s(r) \). To overcome this inconvenience and solve the problem, we could make the hypothesis of a weak scattering object and hence use the two approximations that are the basis of diffraction tomography: the Born and Rytov approximations. They require slightly different assumptions to be valid, but both the resulting forms are similar.

Let us rewrite eq. (1.17) as

\[
\begin{align*}
    u_s(r) &= g(r-r') \rho(r') u_0(r') \, dr' + \int g(r-r') \rho(r') u_s(r') \, dr' \\
    &= - \int \frac{\rho(r') u_0(r')}{|r-r'|} \, dr'.
\end{align*}
\]

(1.18)

where we have resorted to the three-dimensional Green's function \( g(r-r') \).

According to the above mentioned condition of weakly scattered field, \( u_s(r) \) is small when compared to \( u_0(r) \) and the contribution of the second integral in (1.18) can be neglected to achieve the form of the first-order Born approximation:

\[
\begin{align*}
    u_s(r) &\approx u_a(r) = \int g(r-r') \rho(r') u_0(r') \, dr'.
\end{align*}
\]

(1.19)

Better estimates can be found by recursively adding the Born approximation found to the incident field, resulting in the generic \( n \)-order Born approximation:
Linearizing the problem

\[ u^{(n+1)}_a (r) = \int g(r - r') o(r') \left[ u_0 (r') + u^{(n)}_a (r') \right] dr'. \quad (1.20) \]

A necessary requirement for the Born approximation to be valid is that the change in phase between the incident field and the wave propagating through the object has to be less than \( \pi \).

Alternatively, the Rytov approach represents the total field through a complex phase \( \phi(r) \)

\[ u(r) = e^{\phi(r)}. \quad (1.21) \]

In eq. (1.21) \( \phi(r) \) is the sum of the incident phase function \( \phi_0 (r) \) and the scattered complex phase \( \phi_s (r) \). Reconsidering eq. (1.13), we can write:

\[ \left( \nabla \phi(r) \right)^2 + \nabla^2 \phi(r) + k_0^2 = o(r). \quad (1.22) \]

Then, by considering only the scattered component we obtain:

\[ 2 \nabla \phi_0 (r) \cdot \phi_s (r) + \nabla^2 \phi_s (r) = - \left( \nabla \phi_s (r) \right)^2 - o(r). \quad (1.23) \]

Using a plane wave for the incident field, and taking into account the following equation:

\[ \nabla^2 (u_0 (r) \phi_s (r)) = \nabla^2 u_0 (r) \cdot \phi_s (r) + 2 \nabla u_0 (r) \cdot \nabla \phi_s (r) + 
\]

\[ + u_0 (r) \nabla^2 \phi_s (r) \quad (1.24) \]
Linearizing the problem

we obtain:

\[ (\nabla^2 + k_0^2)u_0(r)\phi_s(r) = -u_0(r)\left[ (\nabla \phi_s(r))^2 + o(r) \right] \]  \tag{1.25} \]

whose solution can be expressed by the following integral equation:

\[ u_0(r)\phi_s(r) = \int g(r - r')u_0(r')\left[ (\nabla \phi_s(r'))^2 + o(r') \right]dr'. \]  \tag{1.26} \]

The situation is similar to that found in (1.18) and, unless we operate some approximation in order to linearize the problem, \( \phi_s(r) \) can not be exactly recovered.

By assuming that that:

\[ (\nabla \phi_s(r))^2 + o(r) \approx o(r), \]  \tag{1.27} \]

we obtain the first-order Rytov approximation for the function \( u_0(r)\phi_s(r) \), which is equal to:

\[ u_0(r)\phi_s(r) = \int g(r - r')u_0(r')o(r')dr'. \]  \tag{1.28} \]

The complex phase of the scattered field is therefore expressed as follows:
Linearizing the problem

\[
\phi_s(r) = \frac{1}{u_0(r)} \int g(r - r') u_0(r') \rho(r') dr'.
\] (1.29)

If we suppose a linear relation between the object function and the refractive index

\[
o(r) = k_0^2 n_\delta(r),
\] (1.30)

we obtain

\[
n_\delta(r) \gg \frac{\left(\nabla \phi_s(r)\right)^2}{k_0^2}.
\] (1.31)

With reference to the above considerations, it is worth underlining that, unlike in Born approximation, the size of the object is not a factor in Rytov approximation. In fact, rewriting eq. (1.31) as

\[
n_\delta(r) \gg \left[\frac{\nabla \phi_s(r)}{2\pi} \Delta\right],
\] (1.32)

it is apparent that the change in the scattered phase over one wavelength must be small, and not the complex amplitude of the scattered field.

The presented linear approximation states in a clear way the forward problem and allows us to exploit the equivalent of the Fourier Slice Theorem for X-rays, namely, the Fourier Diffraction Theorem for diffraction tomography. This theorem relates the Fourier transform of the measured forward scattered data with the Fourier transform of the
Linearizing the problem

investigated object under the condition of weakly scattering inhomogeneities. More in detail, it affirms that when an object (Fig. 7) is illuminated with a plane wave $u_0$, the Fourier transform of the forward scattered field measured along a line provides the values of the two-dimensional transform of the object along a semicircular arc in the frequency domain. If the cross-section distribution of the object is considered independent of the position along the $z$ axis and we have to care about a three-dimensional object, we can perform the reconstruction since the forward scattered field has to be measured on a plane.

Figure 7 - The Fourier Diffraction Theorem relates the Fourier transform of a diffracted projection to the Fourier transform of the object along a semicircular arc of ray $k_0$. Illuminations from all possible angles results in a disk formed by the ensemble of all the semicircles.

The radius $k_0$ shown in Figure 7 represents the free-space wavenumber of the incident radiation and, as the frequency increases, it grows generating new estimates of the entire Fourier transform of the object (Figure 8). Therefore a single plane wave provides exact information up to a frequency of $\sqrt{2}k_0$ about the Fourier transform of the object along a
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semicircular arc with radius proportional to the wavelength in the spatial frequency domain. Another procedure to collect more data [17],[18] consists in rotating the object and measuring the scattered field for different orientations. Each orientation will provide a different estimate of the object’s Fourier transform along a circular arc (Figure 8) and these arcs will rotate as the object is rotated.

Figure 8 – Generation of a better estimate of the Fourier transform by changing the direction of the impinging wave and its frequency.

The space domain reconstruction is then executed recurring to the Filtered Backpropagation algorithm [19].
1.3 FURTHER CONSIDERATIONS

To conclude our brief introduction, we need to add some other considerations about the mapping operated in the forward problem, between the collection of the values we want to reconstruct (image) and the quantities we are able to measure (data).

We have already observed the non-linearity of the scattering problem ((1.17) and (1.26)) as an example of the difficulties both analytical and numerical affecting a complete and exact inversion procedure. However, we are also unable in practice to perform exact measurements, because the data can be intrinsically modified by noise or altered by the discretization process produced by a computer representation. Therefore, the forward problem connects the image to error-free data while the data we actually collect are affected by measurements errors. This is another challenge for the inversion consisting of finding the original image given the data and the knowledge of the forward problem. A good insight of this aspect and its implications can be found by considering the conditions provided by Hadamard, under which the inverse problem of finding \( f \) from

\[
Af = d, \tag{1.33}
\]

is well-posed for any given \( d \). In particular, eq.(1.33) is well-posed if it satisfies the following conditions:

1. a solution exists for any data \( d \) in the data space;
2. the solution is unique in the image space;
3. the inverse mapping \( d \rightarrow f \) is continuous.

The first two conditions state that the operator \( A \) has a well defined inverse and that the domain of this inverse is all of the data space, while the last implies that small changes in data lead to small changes on solution. Hadamard defined a problem to be *ill-posed* if not satisfying all of the three conditions and consequently asserted that in this case an inverse does not exist because the data set is outside the range of \( A \), or the inverse is not unique because more than one image are mapped to the same data, or because an arbitrarily small change in the data can cause an arbitrarily large change in the image. The latter aspect can be captured by adding the concept of *ill-conditioning*. If \( \Delta d \) is a variation of \( d \) and \( \Delta f \) denotes the corresponding variation of \( f \), then it can be proved that:

\[
\frac{\| \Delta f \|}{\| f \|} \leq \frac{\sigma_1}{\sigma_N} \frac{\| \Delta d \|}{\| d \|},
\]  

(1.34)

where \( \sigma_1 \) and \( \sigma_N \) are the greatest and the smallest singular values of \( A \), respectively, and their ratio is referred as the *condition number*. Since the fractional error in \( f \) can be equal to the condition number multiplied by the fractional error in \( d \), smaller values of the condition number are desirable. In fact, if the condition number is large, small changes of \( d \) could be responsible of very large changes in the inverse solution. Therefore, a problem is said to be *ill-conditioned*.

The examples shown about inverting weakly scattering objects by linearizing the scattering model have the advantage to be very simple, but they are almost useless since very few materials are characterized by a scattering phenomenon which is weak enough to justify a linear
Further considerations

approximation of the scattering equations [20]. Possible improvements to this kind of approaches are either to iteratively refine the model [21], or to add a few higher-order terms to the series expansion of the scattering operator [22]. Another possibility is to retain the linear model but use a nonlinear reconstruction algorithm, based on possible analytical properties of the object under test [17],[18]. Another approach consists in adopting a fully nonlinear data model in conjunction with properly derived algorithms, this will be discussed in the following chapter.
Chapter 2

Imaging from Scattering Data as an Optimization Problem

2.1 STATEMENT OF THE PROBLEM

The inversion of a full-wave 3D vector scattering problem is challenging not only from a theoretical point of view; it also requires a complex measurement system (i.e. multiview and multifrequency) in order to obtain a data set which contains enough information to perform the 3D analysis. However, these requirements can be sometimes loosened, especially when full imaging is not needed for diagnostic purposes, as often happens in industrial non-destructive testing and evaluation. Moreover, depending upon the simplifying assumptions made, the problem may become scalar and two-dimensional (2-D), or even one-dimensional (1-D). Therefore, if the geometry of the imaging problem can be considered one-dimensional, the data collected by a monostatic, single-view, multifrequency system can contain a sufficient level of information to reconstruct the dielectric range profile of the investigated object.

Among the variety of methods applied to electromagnetic tomography, accurate results can be provided by a fully non-linear data model, in conjunction with properly derived algorithms. This kind of
approach has been followed since the late ‘70s [23], with reconstruction algorithms derived in both the variational [24] and the numerical frameworks [25],[26]. All the reconstruction methods based on variational, numerical or, more recently, stochastic approaches, boil down to optimizing a functional that contains a data-fit part together with a regularization term. The regularization term stabilizes the solutions by enforcing local smoothness in them. Whereas simple regularization strategies [27] normally lead to oversmoothed solutions, we should be allowed to preserve steep slopes or discontinuities where these are likely to occur. Although this is generally paid by a high computational complexity, it has recently become possible by the ongoing increase in computing power, and therefore the first practical microwave computed imaging systems have been implemented, for both clinical [28] and other noninvasive evaluation applications [29], [30].

In this work, it is presented a reconstruction algorithm that is based on a fully nonlinear model and is able to preserve the discontinuities with a reasonable computational effort. At present, we are only considering the case where single-view, multifrequency backscattering data are available, and the reconstruction of a one-dimensional, low-loss permittivity range profile is only required. This is motivated by our interest in non-destructive testing of building materials, where the simplicity of the measurement system is an important feature. Moreover, reconstructing one-dimensional permittivity profiles is a problem with its own interest in several fields, such as ground-penetrating-radar signal processing and non-destructive diagnosis of lapideous artifacts for cultural heritage monitoring. The availability of a number of 1D range profiles, on the other hand, allows a 2D image to be reconstructed by spherical backprojection [31].
The typical relative permittivities encountered in these applications are of the order of 10 (for example, the relative permittivities of concrete lie in the range 8-12). Strong and abrupt inhomogeneities are very likely to occur, for the presence of voids, inclusions of heterogeneous materials, porosity, or zones with variable water content. This means that, in our cases, a non-linear data model is absolutely needed and the regularization strategy must take discontinuities into account.

Our strategy consists in adopting a regularization term that explicitly includes the discontinuities as unknowns of the problem [31]. More specifically, our model for the solution consists of a pair of interacting Markov random fields. One is continuous, and represents the permittivity values of the tested object. The other is binary, and its status denotes presence/absence of a discontinuity between two adjacent sites of the permittivity random field. The reconstruction is then performed as a Maximum a Posteriori (MAP) estimation, where, for the presence of the discontinuity field, the posterior probability function is non-convex. This requires a global optimization strategy and, since evolutionary computing can be used to implement excellent global optimizers, we choose to solve the problem by employing both genetic algorithms as well as particle swarm intelligence.

In the following two sections, we introduce our specific energy function and describe the forward solver used to deal with the forward problem; then we turn our attention to the inversion procedure of the data through genetic and particle swarm optimization algorithms.
2.2 THE REGULARIZATION PROCEDURE

In this section, we first describe the problem of scattering imaging in a 2D setting, and then particularize it to our 1D application. We show how the problem reduces to an optimization and which is the form of the objective function.

Let us assume to deal with a dielectric body that can be characterized by a 2D permittivity distribution, immersed in a homogeneous environment and illuminated by a known electromagnetic field. In general, both the frequency and the incidence direction of the probing wave are variable (multiview-multifrequency illumination system). The scattered field is coherently measured by a number of sensors placed in a fixed measurement domain. The sensor positions characterize a set of scattering directions for each incidence direction (multistatic measurement system).

Our problem is now to reconstruct the permittivity as a function of space, on the basis of all the scattering data collected by the measurement system. Let us assume that we are able to evaluate the scattered field at all the sensors from the knowledge of the incident field and the permittivity function \( \varepsilon(\mathbf{r}) \) (where \( \mathbf{r} \) is the 2D position vector). If the permittivity function is unknown, we can devise a strategy to estimate it as the one that reproduces the actual scattered field. In formulas, if \( E_{\text{meas}}(\mathbf{r}) \) is the scattered field measured at any \( \mathbf{r} \) in the measurement domain and \( E_{\text{calc}}(\mathbf{r}, \varepsilon) \) is the scattered field calculated for a tentative permittivity function \( \varepsilon(\mathbf{r}) \), our permittivity estimate, \( \hat{\varepsilon}(\mathbf{r}) \), could come from the following optimization:
The regularization procedure

\[ \hat{\varepsilon}(r) = \arg \min_\varepsilon d \left( E_{\text{meas}}(r), E_{\text{calc}}(r, \varepsilon) \right), \quad (2.1) \]

where \( d \) is a suitable distance. The above relation only exploits a data-fit criterion to reconstruct the permittivity distribution. Of course, matching calculated and measured data is a necessary condition to obtain a good reconstruction, but this does not imply that it is always sufficient. Indeed, eq. (2.1) is never a sufficient condition to provide a unique solution to the reconstruction problem [1],[11]. In fact, since there is always some kind of information loss which affects the data collected from the measurements, if we want to be able to successfully find an acceptable inverse solution we need to use additional constraints on the problem. This does not mean that we retrieve the lost information, but that some additional information is normally needed to render the solution unique. The regularization consists in incorporating some piece of available prior knowledge in the problem, thus overcoming its ill-posedness. All the linear approaches to both diffraction and projection tomography implicitly assume that the image function is band-limited, therefore introducing some degree of smoothness in the solution. In the framework set by eq. (2.1), the problem can be regularized by introducing additional terms in the objective function:

\[ \hat{\varepsilon}(r) = \arg \min_\varepsilon d \left( E_{\text{meas}}(r), E_{\text{calc}}(r, \varepsilon) \right) + \lambda U(\varepsilon). \quad (2.2) \]

In particular, function \( U \) is the regularization potential, which does not depend on the data but only expresses prior information on the solution. The weight \( \lambda \) is the regularization coefficient and introduces a
The regularization procedure compromise between the data and the prior knowledge. Equation (2.2) is the most general form of image reconstruction reformulated as an optimization problem. The differences among the various approaches adopted in the open literature lie in the different distance measurements and regularization potentials employed. As mentioned above, the regularization potentials express some degree of smoothness in the solution, that is, a limitation on the magnitudes of $\varepsilon(\mathbf{r})$ and some of its derivatives. When $U$ is independent of location, the magnitudes of the derivatives are constrained to be small everywhere. Thus, possible discontinuities in the function to be reconstructed are certainly lost (i.e., the solution is \textit{globally} smooth). Since discontinuities carry essential information on an image, this is an undesirable effect, and should be avoided. This is normally achieved by building a regularization potential that can relax its smoothing effect where the solution is likely to be discontinuous (the solution will be \textit{locally} smooth). Many solutions have been proposed in these years. Among them, we focus on the explicit-line Markov random field approach [31], where the discretized image is modeled as a pair of interacting Markov random fields, a continuous one representing image intensity and another (binary valued) explicitly marking the locations where the solution can be discontinuous. In this approach, and if we only want to constrain the first partial derivatives of $\varepsilon(\mathbf{r})$, the regularization potential in (2.2) becomes:

$$U(\varepsilon, l) = \sum_{k \in A} \sum_{d=1}^{2} \left| D_d(k) \right|^2 (1 - l_{dk}) + \alpha l_{dk}, \quad (2.3)$$

where $\varepsilon$ is a pixelized version of the function to be reconstructed, $l$ is a binary-valued line process located in the interpixel sites of the image.
grid $A$, $D_d(k)$ are discrete approximations to the two first partial derivatives at pixel $k$, and $l_{dk}$ (the elements of the line process) are binary values that break the smoothness constraint across any pair of pixels when their value equals one. Indeed, it is easy to see that when a line process element is unity, the related part of the regularization potential assumes a constant value $\alpha$, independently of the magnitude of the first derivatives at that point. Note that the quantity $\sqrt{\alpha/\lambda}$ acts as a threshold on the value of $|D_d(k)|$ allowing a discontinuity to be introduced. The potential is smaller for $l_{dk}=1$ when this threshold is exceeded, and for $l_{dk}=0$ otherwise. When the regularization term has the form in (2.3), the functional in (2.2) is non-convex for the presence of the line process, and a global optimization algorithm is required.

Let us now identify the form assumed by the objective functional in our 1D case. The measurement geometry for our application is shown in Figure 9.

**Figure 9** – Measurement setup for a normal-incidence, monostatic interrogation system of a multilayered structure.
The regularization procedure

The object under test is a lossless wall of total thickness $L$, immersed in air, whose permittivity only depends on a depth coordinate $z$. This structure is probed by a normally incident, microwave, frequency swept, plane-wave radiation. For a number $N_f$ of discrete frequencies within the system bandwidth, we measure the complex reflection coefficients at the air-wall interface:

$$\rho_{\text{meas}}(f_i) = \frac{E_s(z_0, f_i)}{E_i(z_0, f_i)},$$

where $E_s$ and $E_i$ are the backscattered and incident fields, respectively, $z_0$ is the depth coordinate of the air-wall interface (see Figure 10), $f_i$ is the generic measurement frequency, and no depolarization effects are considered. The imaging problem outlined above now consists in estimating the permittivity as a function of $z$, from the complex $N_f$-vector $\rho_{\text{meas}}$ of the reflection coefficients.

**Figure 10** - Transmission-line equivalent of the investigated wall discretized into a finite number of homogeneous layers.
The regularization procedure

The discrete model of this experiment is described in Figure 10. The wall is divided into a finite number $N$ of homogeneous layers. Each layer is assigned with a uniform permittivity $\varepsilon_{i+1} = \varepsilon \left[ (z_i - z_{i+1})/2 \right]$. The real $N$-vector $\varepsilon$ thus formed is a discretization of the permittivity profile. In this case, the functional in (2.2), with the regularizer in (2.3), becomes

$$F(\varepsilon) = d \left( \rho_{\text{meas}}, \rho_{\text{calc}}(\varepsilon) \right) + \lambda \sum_{k=1}^{N-1} (\varepsilon_{k+1} - \varepsilon_k)^2 (1 - l_k) + \alpha l_k,$$  \hspace{1cm} (2.5)

where the forward differences were used to approximate the first derivatives. The generic line element, $l_k$, is a binary (0–1) variable whose value affects the allowed difference between the $k$-th and the $(k+1)$-th permittivity values.

Any algorithm aimed at minimizing the functional $F$ must be able to evaluate the complex $N_f$-vector $\rho_{\text{calc}}$ for any real $N$-vector $\varepsilon$, to evaluate the distance $d$, and have a strategy for choosing the parameter $\lambda$. For obvious reasons, the computational tool to evaluate the vector $\rho_{\text{calc}}(\varepsilon)$ is referred to as the forward solver. Basically, to guarantee consistency and efficiency, accuracy and a low computational cost are required. These opposite needs often lead to a compromise solution. In our work, we adopted a solver that treats the material layers as a cascade of $N$ transmission line segments, with lengths equal to the thicknesses of the layers and characteristic admittances determined by the related elements of the vector $\varepsilon$. The overall reflection coefficient is determined by the values of the admittance seen at the interface $z = z_0$ and the free-space characteristic admittance.
The regularization procedure

Let $\beta_n$ be the propagation constant in the $n$-th layer ($\beta_0$ in free space) at a generic frequency. The characteristic admittance of the $n$-th equivalent transmission line segment is $\omega \varepsilon_n / \beta_n$. We set the admittance $Y(z_N^-)$ seen at the interface $z = z_N$ towards the negative $z$ axis, at the value of the characteristic admittance of free space:

$$Y(z_N^-) = \frac{\omega \varepsilon_0}{\beta_0}. \quad (2.6)$$

Then, for each interface at $z_n$, with $n$ decreasing from $N$ to 0, we repeat the following steps:

1) Compute the normalized admittance seen through the interface towards the negative $z$ axis:

$$y(z_n^+) = \frac{Y(z_N^-)}{\omega \varepsilon_n / \beta_n}; \quad (2.7)$$

2) Compute the reflection coefficient $\rho(z_n^+)$ at the interface $z = z_n$:

$$\rho(z_n^+) = \frac{1 - y(z_n^+)}{1 + y(z_n^+)}; \quad (2.8)$$

3) Rotate the reflection coefficient to the next interface:

$$\rho(z_{n-1}^-) = \rho(z_n^+) e^{-j\beta_n (z_{n-1} - z_n)}; \quad (2.9)$$

4) Compute the unnormalized admittance at the next interface:
The regularization procedure

\[ Y(z_{n-1}^-) = \frac{\omega \varepsilon_n}{\beta_n} \frac{1 - \rho(z_{n-1}^-)}{\rho(z_{n-1}^-)} ; \quad (2.10) \]

v) Set \( n = n-1 \) and repeat the procedure from step i).

For \( n=0 \), step (ii) provides the value of the reflection coefficient at the interface \( z=z_0 \) at the angular frequency \( \omega \). After repeating the whole procedure for all of the frequencies, we obtain the \( N_f \)-vector \( \rho_{\text{calc}}(\varepsilon) \). Vector \( \rho_{\text{calc}}(\varepsilon) \) can then be used in eq. (2.5) to evaluate the objective function and, as mentioned above, the presence of the line process renders this function non-convex.

Since the combined synergy of a forward solver and an optimization method has proved to be a reliable and efficient approach in the electromagnetic context, we adopted as an optimization tool both genetic algorithms [32]-[34] and particle swarm optimization [35]-[37]. They are both nondeterministic procedures that provide some advantages over both deterministic and other stochastic algorithms. Indeed, unlike most deterministic algorithms, they are normally independent of the starting point in the solution space. Also, unlike other stochastic algorithms such as simulated annealing, their approach does not need to sample from any probability density related to the objective function. All nondeterministic approaches to optimization are quite expensive computationally, however, when problems are complex, these are competitive with deterministic algorithms and show several levels of parallelism that can be exploited.

In the next section, we will briefly introduce the main concepts of genetic algorithms optimization and we will describe the paradigm of particle swarm intelligence.
2.3 OPTIMIZATION METHODS

Genetic Algorithms (GAs) are global stochastic search methods that follow the metaphor of biological evolution [38]. They are patterned after the natural processes of genetic heredity and variation in organisms. GAs operate on a set of potential solutions (population) employing the Darwinian principle of survival of the fittest to produce approximations to a solution. These approximations (individuals) are encoded as strings (chromosomes) using various alphabets (binary, integer, real). The chromosome values represent genotypes and are mapped onto the decision variable (phenotypic) domain, but the search process is carried out on the encoded variables rather than on the decision variables. In order to assess the performance of members of a population, chromosomes are decoded to obtain a representation into the solution space. The goodness of a chromosome is evaluated through an objective function that gives a value (fitness) related to the individual’s aptitude for the current problem. In biological terms, this number expresses to the capability to survive in a given environment. A new set of approximations is created at every generation by selecting individuals according to a selection rule and breeding a new generation by using the cross-over and mutation operator. This process drives towards individuals better than the elder ones just as in natural adaptation. The fitness function is generally non-negative defined and its minimum value is generally considered as the optimum. Therefore the process referred to as “fitness optimization” is a minimization procedure. GAs achieve their best results in exploring multi-dimensional and multi-modal function domains in order to reach an
approximate global optimum. They are largely independent of the initial conditions and place few constraints on the solution domain. Thanks to these characteristics, GAs are able to deal with solution spaces having discontinuities and a large number of dimensions with many local fitness optima, as in the inverse scattering problem we have to face.

In our case, the relevant parameters are a set of $N$ real permittivity values plus a set of $N-1$ binary line element values. Each parameter is mapped onto the chromosome as a substring called a gene, and the chromosomes are organized into populations of constant size. Each chromosome is used to evaluate $\rho_{\text{calc}}$ by the procedure (2.6)-(2.10), and the related value of the objective function, by eq. (2.5). As said before, the objective function is used to evaluate the quality of the proposed solutions, and to determine, stochastically, which of them will breed the successive generations. By using the crossover operator, a randomly chosen pair of chromosomes (the parents) generate a pair of children chromosomes. Each child inherits a different subset of genes from each parent. On the other hand, mutation is also employed and it basically consists in randomly switching a single bit in each chromosome string, and has the effect of enhancing the capability of exploring the solution space, thus enabling the procedure to avoid local optima. To assure the attainment of a global optimum, the strategy of simple elitism is adopted. This consists in replacing the worst individual of the current population by the best of the previous one, if the latter has a better fitness than the best of the current population. The overall procedure is repeated throughout the generations until no further improvement is observed in the fitness.

The Particle Swarm Optimization (PSO) is an evolutionary computation technique inspired originally proposed by Kennedy and
Optimization methods

Eberhart [39]. The most famous image that recalls the intrinsic characteristics of the PSO is that of a swarm of bees in a field [40]. The goal of the swarm is to find out the location with the most attracting flowers. Without any a priori knowledge, the bees start to look for the flowers with random trajectories and velocities. It is supposed that each bee is able to remember the location where it found the most flowers and that it also knows the information gained by the other bees about the best places they have found. The bee is therefore attracted both by the location where it had personally found the most flowers and by the place reported by the others to be the most promising position. As a result, the bee wanders somewhere between these two points and can reach an unexplored place. The process goes on until all the bees converge on the best location.

Aside from the natural analogy, in PSO each bee (or agent) that traverses in the multidimensional space represents a possible solution of the investigated problem. Each dimension of this space represents a parameter of the problem to be optimized. During the search procedure that seeks the best location in the solution domain, the agents change their position with time as they fly around in a multi-dimensional solution space. As previously mentioned, during their flight, particles adjust their velocity according to their own experience and those of the other particles, making use of the best position reached, both by themselves and by the rest of the swarm. Hence, the agent is stochastically attracted both toward the best personal as well as toward the best overall location, while evaluating the goodness of its current location in the process.

More in detail, each particle of the swarm in the \( n \)-dimension space has a position at a certain instant \( i \) which is identified by the vector of the coordinates \( X \)
Each $x_n(i)$ component represents a parameter describing our physical problem. At the beginning of the process, each particle is randomly located and moves with a random velocity, both in direction and magnitude. The particle is free to fly inside the $n$-dimensional space defined by the user, within the constraints of the $n$ boundary conditions that limit the extent of the search space and, hence, the values of the parameters during the optimization process. At the generic time step $i+1$, the velocity is expressed by the following equation

$$v_n^k(i+1) = w * v_n^k(i) + c_1 * \text{rand()} * (p_{\text{best},n}^k(i) - x_n^k(i)) + c_2 * \text{rand()} * (g_{\text{best},n}^k(i) - x_n^k(i))$$

(2.12)

where $v_l(i)$ is the velocity along the $l$ direction at time step $i$, $w$ is the inertial weight, $c_1, c_2$ are constants expressing the cognitive and the social rate, respectively, $p_{\text{best},l}(i)$ is the best position along the $l$ direction found by the agent during its own wandering up to $i$-th, $g_{\text{best},l}(i)$ is the best position along the $l$ direction discovered by the entire swarm; and $\text{rand()}$ is a generator of random numbers uniformly distributed between 0 and 1.

At each iteration step, the new velocity is the sum of the actual velocity, scaled by the factor $w$, which represents the weight of the particle, and two terms that express the attraction due to $p_{\text{best}}$ and $g_{\text{best}}$ (Figure 11).
Figure 11 - Example for the optimization of three parameters. Each agent is attracted both by its own best location (grey circle) and by the best place found by the entire swarm (black star). The resulting velocity (black arrow) forces the agent to explore a new position.

The first of these two terms determines how much the agent is influenced by the memory of its own best (referred before as the “cognitive rate”) and the value of $c_1$ encourages the independent search for the best, regardless of the experience of the swarm. On the other hand, the second term is related to the influence the swarm has on the particle (called the “social rate”) and $c_2$ controls the exploitation of the actual best. The random generator introduces the proper chaotic component of a real swarm. The position of each particle is then simply updated according to the equation:

$$x^k_{n}(i + 1) = x^k_{n}(i) + v^k_{n}(i) \Delta t$$

(2.13)

where $x_l(i)$ is the current position of the agent along the direction $l$ at the iteration $#i$, and $\Delta t$ is the time step. The initialization and position update
Optimization methods require some further considerations because they can considerably change the overall performance of the algorithm. When a particle, during its wandering in the solution domain exceeds the allowed range for one parameter, it needs to be properly treated by employing some boundary condition. It can be prevented from assuming a value greater than allowed by zeroing the velocity along that dimension and stopping the particle right at that boundary (absorbing wall condition). It can be reflected along that direction, just by turning in the opposite direction the velocity (reflecting wall condition). Or it can just be let passed through the wall but without considering it when the fitness value is calculated (invisible wall condition). The flow chart depicting the PSO algorithm is shown in Figure 12.

![Flow chart of the PSO algorithm](image)

**Figure 12** - Flow chart of the PSO algorithm: after the initialization, for each particle the fitness is scored and the values of the local and overall best are updated. Then, if the desired value for the fitness has not been satisfied the process goes on by updating the velocity and position, assuring that all the agents are within the required set.

All the boundary conditions presented above affect the convergence rate depending upon the specific problem. We have developed an interesting
Optimization methods

and promising new method to initialize and update the positions, which has shown to be reliable and efficient.

Much effort has been invested towards improving the performance and the reliability of the PSO. The problems of PSO parameter control and tuning have been widely investigated in [41]-[45] and the effects of changing the neighborhood topology have been discussed extensively in [46]-[51]. To the best of our knowledge, however, position initialization, and its effects on the convergence rate have not been fully addressed yet. Since, in most cases, the agents are randomly located, especially in high-dimensional spaces, it may happen that some areas have higher densities of particles than others. Of course, this inhomogeneity does not prevent the agents from pursuing their goal, but can affect the time required for reaching the final solution.

The solution we have proposed [52] to circumvent this difficulty is to subdivide the solution space into sub-domains within which groups of agents are initially located. These sub-domains are defined by sub-boundary conditions that are derived from the original boundaries, and their presence guarantees the homogeneous distribution of agents all over the search domain. Each particle only cooperates with those particles in its own group independently from the other groups, ensuring an improvement of exploration of their part of the global domain defined by the sub-boundaries. Then, after a fixed number of iterations, the sub-boundaries are removed, the best positions found by each group are scored and the actual global best location is revealed to all. It is demonstrated that the first part of the optimization process, managed by particles inside the sub-boundaries, improves the speed to find the optimal solution and hence increases the convergence rate of the process. To help understanding the novelty introduced by the initialization of the sub-
Optimization methods

boundaries, we first report, in Figure 13, a pseudocode for the standard particle swarm optimization.
Define the allowed range of each dimension (boundaries)
Set $i=1$

for $k=1$, number_of_agents
  for $n=1$, number_of_dimensions
    Random initialization of $x_k^n(i)$ within the allowed range $[x_{n,min}, x_{n,MAX}]$
    Random initialization of $v_k^n(i)$ proportional to the dynamic of dim. $n$
  next $n$
next $k$

for $j=1$, number_of_iterations
  for $k=1$, number_of_agents
    Evaluate $fitness^k(i)$, the fitness of agent $k$ at instant $i$
  next $k$

Rank the fitness values of all agents

for $k=1$, number_of_agents
  if $fitness^k(i)$ is the best value ever found by agent $k$ then
    $p_{best,k}^i(i) = x_k^n(i)$
  end if
  if $fitness^k(i)$ is the best value ever found by all agents then
    $g_{best,a}^i(i) = x_k^n(i)$
  end if
next $k$

Update agent velocity by using (2) and limit if required
Update agent position by using (3), check it with the Boundaries

$i=i+1$
next $j$

Figure 13 – PSO implementation with initialization by using boundary conditions.
Our solution has its basis on the simple observation that there exists a high probability that the initial step of randomly positioning all the agents can determine a non-uniform coverage of the search domain. This fact affects the convergence rate, especially if the domain is large compared to the number of agents involved in the search process.

Even if the algorithm is able to find the optimal solution, the process could be sped up by adopting an approach that will be detailed in this section. The underlying concept which the algorithm is based upon is to uniformly distribute the agents at the start of the optimization process. The agents are organized into equal groups and these groups are then forced to exploit a sector of the domain defined by the sub-boundaries. This concept is described in Figure 14 for a three-dimensional domain.

![Figure 14](image)

**Figure 14** - The domain defined by the boundaries is split into sectors defined by sub-boundaries within groups of agents wandering in search of the best location. The sub-division could play an important role in finding faster the best solution (asterisk).

The domain is subdivided into sectors (or sub-domains) by using sub-boundaries that split one or more dimensions into equal intervals. The number of sub-boundaries cannot exceed the number of agents but, as it will be apparent later, should not produce groups that are too small. During the initial stages, each group flies inside the assigned sub-domain...
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and, hence, each group $g$ has its own “sub-domain best” (indicated by $g_{best,n}^g$). Furthermore, each agent $k$ in the group $g$ has its own local best location ($p_{best}^{k,g}$). The sub-boundaries pose impassable limits and, consequently, none of the agents of one group can pass through these boundaries and enter another sector. This guarantees that the number of agents in each sector is constant and that the homogeneity of their spread within the multidimensional domain is preserved. Once the number of iterations dedicated to this process is exceeded, the barriers imposed by the sub-boundaries are removed and the particles are free to fly all over the entire domain. The “global best” is then chosen from those found in the sectors by the groups while the “local best” position of each agent is preserved. The operation executed at the exact instant of the passage from the sub-boundary conditions to the global boundary conditions is described in Figure 15 and Figure 16 for a two-dimensional case. To highlight the differences introduced in this modified version of the PSO, its pseudocode is presented in Figure 17.
Figure 15 - After the last iteration in sub-domain mode, and before starting the entire domain discovery, each particle is attracted by its own sub-domain best (black dots) and its local best (grey squares). The star in sector 2 is the best of all the sectors’ bests.

Figure 16 - Dynamics of a change: all the agents gain the information about the global best as soon as the barriers imposed by the sub-boundaries are removed. They are attracted both by that location as well as by the own local best previously found.
Optimization methods

To check the improvements obtained by this new PSO implementation, we have optimized several functions used as test beds for studying the performance of optimizers [40],[41]. In particular, the following functions have been considered. The first type is the Rastrigin function, defined as

$$f_1(x) = \sum_{i=1}^{N} \left( x_i^2 - 10 \cos(2\pi x_i) + 10 \right)$$

with (-5.12 < $x_i$ < 5.12).

The second type is the Griewank function (-600 < $x_i$ < 600):

$$f_2(x) = \frac{1}{4000} \sum_{i=1}^{N} x_i^2 - \prod_{i=1}^{N} \cos \left( \frac{x_i}{\sqrt{i}} \right) + 1$$

Last function is the Rosenbrock function

$$f_3(x) = \sum_{i=1}^{N} \left( 100 \left( x_{i+1} - x_i^2 \right)^2 + (x_i - 1)^2 \right)$$

with (-50 < $x_i$ < 50). All the above-introduced functions have a global minimum equal to zero. Several simulations have been run for each of these functions, by both the standard and the modified PSO algorithms. Three different swarm sizes have been considered, comprising 16, 20 and 32 agents, respectively. Furthermore, to better understand the influence of the sub-boundary initialization, we have addressed the problem with a variable number of sectors (2, 4, and 8) and, hence, different number of groups (note that, as previously mentioned, each sector contains only one group).
Set $i=1$
for $g=1$, number_of_groups
  for $k=1$, number_of_agents_in_the_group
    for $n=1$, number_of_dimensions
      Random initialization of $x_n^{k,g}(i)$ within the range of subdomain $#g$
      Random initialization of $y_n^{k,g}(i)$ proportional to the dynamic of subdomain $#g$
      next $n$
    next $k$
  next $g$
do while (Sub_boundary_case)
  Flag_set_global_best = FALSE
  for $g=1$, number_of_groups
    for $k=1$, number_of_agents_in_the_group
      Evaluate $\text{fitness}^{k,g}(i)$, the fitness of agent $k$ in group $g$ at instant $i$
    next $k$
  next $g$
  for $g=1$, number_of_groups
    Rank the fitness values of all agents included only in group $g$
  next $g$
  for $g=1$, number_of_groups
    for $k=1$, number_of_agents_in_the_group
      if fitness$^{k,g}(i)$ is the best value ever found by agent $k$ in group $g$ then
        $g_n^{k,g}(i) = x_n^{k,g}(i)$
      end if
      if fitness$^{k,g}(i)$ is the best value ever found by all agents then
        $g_n^{g}(i) = x_n^{k,g}(i)$
      end if
    next $k$
  next $g$
  $i=i+1$
(continue …)
Optimization methods

(… continued)

```plaintext
if (i >= sub_boundaries_iterations) then
    Sub_boundary_case = FALSE
end do
if (Flag_set_global_best = FALSE) then
    Flag_set_global_best = TRUE
    Rank all the \( g_{best,p}^{g} \) and set the actual \( g_{best,p}^{a} \)
else
    follow, with the actual value of \( i \), the procedure showed in Fig.1
end if
```

**Figure 17** - The modified PSO pseudocode. During the preliminary iterations the agents seek together, organized in groups, in a delimited territory defined by the sub-boundaries. After this stage, they are free to move all over the domain.

The maximum allowed number of iterations to reach the minimum has been set to 150. Except for the boundary case, we have run half of the iterations with active sub-boundaries. This choice has to be considered a limit that should not be exceeded, so as not to lose the efficient cooperation of all the agents acting together, which is one of the most important features of PSO. The results for \( N=3 \) are shown in Table II. The first value expresses the average number of iterations necessary to approach the minimum with a tolerance of less than 0.01. The abbreviation N.R. (not reached) means that this requirement has not been satisfied up to the 150\(-th\) iteration. The second value within brackets is the number of fitness evaluations, which equals the calls to the solver. For the case of 20 agents and 8 sectors we have not proceeded since it is not possible to create groups with the same number of agents. From the above results, it is possible to state that the initialization with the sub-boundaries not only helps us to speed-up the convergence, but also that the more we
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increase the number of divisions the less we improve performance. This fact suggests a logical conclusion, viz., that there is a limit to the improvement even if we subdivide more and more. Of course, the number of sectors is also limited by the number of agents, since a group has to be composed at least by two agents.

### TABLE II
RESULTS OBTAINED BY USING SUB-BOUNDARIES INITIALIZATION IN SOLVING BENCHMARK FUNCTIONS

<table>
<thead>
<tr>
<th>Population size</th>
<th>Boundaries</th>
<th>Rastrigin function</th>
<th>Griewanck function</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Number of groups: 2</td>
<td>Number of groups: 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Number</td>
<td>Number</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Boundaries ()</td>
<td>Boundaries ()</td>
</tr>
<tr>
<td>16 agents</td>
<td>N.R.</td>
<td>142 (2272)</td>
<td>117 (1872)</td>
</tr>
<tr>
<td>20 agents</td>
<td>N.R.</td>
<td>110 (2200)</td>
<td>70 (1400)</td>
</tr>
<tr>
<td>32 agents</td>
<td>N.R.</td>
<td>60 (1920)</td>
<td>40 (1280)</td>
</tr>
<tr>
<td>16 agents</td>
<td>N.R.</td>
<td>122 (1952)</td>
<td>110 (1760)</td>
</tr>
<tr>
<td>20 agents</td>
<td>N.R.</td>
<td>96 (1920)</td>
<td>84 (1680)</td>
</tr>
<tr>
<td>32 agents</td>
<td>N.R.</td>
<td>80 (2560)</td>
<td>52 (1664)</td>
</tr>
</tbody>
</table>
As a final consideration, we have to remark that the GA we used encodes all the parameters in binary strings. Therefore all the quantities such as the dielectric permittivity of a layer as well as the line process are transformed into bits. In the case of PSO, the solution domain is characterized by real parameters, therefore, to include the binary line elements in the model, we also introduced the possibility to treat real and binary parameters simultaneously. A discrete binary version of the PSO was first introduced by Kennedy and Eberhart [53], in which the concept of velocity loses its physical meaning and assumes the value of a probability instead. Specifically, the position along a direction can now be either 0 or 1, and the velocity represents the probability of change for the value of that bit. In light of this, it becomes necessary to modify the expression in (2) by imposing the condition that the value of $v_n(i)$ must be in the interval [0.0, 1.0], and insisting that any values outside this interval be unacceptable. As a consequence, a function ($T$) is needed to map the results of (2.12) within the permissible range. If we let $w=1$ and $c_1=c_2=2$, $v_n(i)$ is in the interval [-4, 5]. The $T$ function linearly compresses this
dynamic range into the desired set [0, 1] and then the position is updated by using the following rule:

\[
\begin{align*}
\text{if } (\text{rand}(\cdot) < T(v_n(i)) \text{ then} \\
x_n(i) &= \overline{x_n(i)} \\
\text{else} \\
x_n(i) &= x_n(i)
\end{align*}
\]

(2.17)

where \( \text{rand}(\cdot) \) is the same random function adopted in (2.12) and the operator \( \overline{x_n} \) indicates the binary negation of \( x_l \). This implies that if the random number is less than the probability expressed by the velocity, the bit is changed. Thus, the faster the particle moves in that direction, the larger is the possibility of change.
CHAPTER 3

Validation and Results

3.1 TEST BED

In this section, we describe the results of our estimation of the algorithm performance. Since the reconstruction accuracy is related to the quality of the forward solver, we first verified the effectiveness of this module by comparisons with the results of commercial simulation codes and direct field measurements. Subsequently, by means of both simulated and real experiments, we evaluated the performance of the reconstruction algorithm in terms of accuracy in discontinuity location, permittivity estimation, spatial resolution and robustness against noise. Assuming to apply our method to nondestructive evaluation of masonry structures, the permittivity values to be treated are in the range 1-13. The working frequency range we are considering is 800 MHz – 3 GHz, typical of many ground-penetrating radar systems. All our numerical simulations have assumed this range. The real experiments, however, have been limited to the range 1.7-2.6 GHz, since this is the band of our available sensor. The setup to validate the solver on real measurements is depicted in Figure 18. The probe/sensor element is a W-band horn antenna connected
to a computer-controlled vector network analyzer. The probed samples are dielectric slabs of size $100\text{cm} \times 100\text{cm} \times 1\text{cm}$, positioned to assure a nearly plane wave incidence.

![Diagram of the experimental setup](image)

**Figure 18** - Layout of the experimental setup

All the measurements were subtracted by the residual background signal and normalized to corresponding measurements made on a copper plate of the same size as the objects under test.
3.2 FORWARD SOLVER VALIDATION

First of all, in order to test the solver, we have verified its reliability by comparing the results obtained for several different structures with the corresponding results obtained through standard commercial electromagnetic simulation software, such as HFSS. As an example, in Figure 19 it is shown a configuration composed by five layers.

**Figure 19** - Configuration of the simulated structure. The PML conditions are applied to prevent spurious reflection inside the computational domain but does not affect the adherence of the simulation to real measurements.

The comparison between the result provided by the commercial software for this structure are compared to those found by our solver is reported in Figure 20. It is interesting to notice that we are comparing results obtained from two different computational methods: the former is based on a Finite Element Method (FEM) while the latter is a transmission line equivalent model.
Figure 20 - Comparison between the magnitude (a) and the phase (b) of the reflection coefficients obtained by probing with a plane wave the structure composed of four layers with permittivities $\varepsilon_r = 2.8$, $\varepsilon_r = 4.0$, $\varepsilon_r = 6.0$, $\varepsilon_r = 1.0$, $\varepsilon_r = 3.0$ and thickness equal to 1.0 cm, 1.0 cm, 3.0 cm, 1.0 cm and 2.0 cm, respectively. The working frequency range is from 0.8 GHz to 3.0 GHz. The incidence of the plane wave is normal. The results obtained by our solver are indicated with the solid line while the dots represent the solution given by Ansoft HFSS.

The good agreement with the result produced by the full wave code proves the reliability of the transmission line model adopted. We have also validated our forward solver with real data.
Forward solver validation

![Graphs showing magnitude and phase of reflection coefficient](image)

**Figure 21** - Comparison between the magnitude (a) and phase (b) of the simulated (black) and measured (gray) reflection coefficient for a single slab of plexiglas.

A first test has been performed on a single plexiglas slab with relative permittivity equal to 2.5. The results, compared to the ones calculated by our solver, are shown in Figure 21. Also in this case, the good agreement between numerical results and the measurements is apparent. Another test has been performed on a three-layer object, composed of a styrofoam slab between two plexiglas slabs, in order to simulate an air inclusion. The
comparison between the predicted numerical results and the real measurements is shown in Figure 22. These results highlight the good agreement between simulated and measured data in the case of multilayered structures and demonstrate the effectiveness and reliability of our solver, at least on this reduced frequency range and these permittivity values.
Figure 22 - Comparison between the magnitudes (a) and phase (b) of the simulated (black) and measured (gray) reflection coefficients for a plexiglas-styrofoam-plexiglas sandwich.
3.3 EXPERIMENTAL VALIDATION

After the validation of the forward solver by using both synthetic and real measurements we adopted the same procedure to assess our inversion procedure. For each experiment, the data consisted in the reflection coefficients measured at 100 different frequencies within the working range. Using synthetic data allows us to test the reliability of the results from structures with exactly known permittivity profiles, that could be difficult to obtain in practice. Another advantage is that the noise levels can easily be controlled and their effects separated from the model errors. Our simulations were intended to test the dynamic range of our reconstruction, and the effectiveness of the line process in improving the spatial resolution. Hereafter, we report some of the most significant results.

For our simulations, we adopted dielectric profiles that describe concrete walls with air inclusions of different thicknesses and differently spaced from each other. The experimental conditions were made more realistic by corrupting the simulated data by variable amounts of white Gaussian noise. Moreover, although the simulation has been performed by resorting to the forward solver used for the reconstruction, the discretization grids used in the two phases are generally different.

At a first stage, we considered the effect of the line process. We observed that the global smoothness enforced by a regularization term without a line process, besides affecting the resolution, as obvious, also renders the permittivity estimation much less accurate. In Figure 23, we compare the results obtained with and without a line process for a
Experimental validation

24 cm-thick wall, with \( \varepsilon_r = 10 \) and an air inclusion inside. As apparent, the dotted line representing the lineless reconstruction seriously underestimates the material permittivity while failing to detect the presence of the void layer. By using the line process, we correctly detected and located air inclusions as thin as 8 mm with SNRs as low as 10 dB.

**Figure 23** - Numerical simulation. Comparison between the actual range profile (solid line) and the permittivity profiles reconstructed by our algorithm with the line process inactive (dotted line) and active (chain line).

The spatial resolution has been evaluated by reconstructing profiles with two air gaps placed at decreasing distances. To put the inclusions closer and closer to each other, we reduced the total wall thickness maintaining a constant number of layers. Also in this case, we found that two 8 mm inclusions, 8 mm apart from each other are clearly detected and distinguished, if the SNR is not smaller than 10 dB. A first example of reconstruction is shown in Figure 24, where the different values of permittivity are recovered with a good accuracy. In Figure 25, we show a
result with 10 dB SNR, where it can be noted that the reconstruction error is still comparable with the 25 dB-SNR cases.

Figure 24 - Numerical simulation. Actual (solid line) and reconstructed (dashed line) range profiles for a 12-cm-thick wall divided into 12 layers. Frequency range 0.8 GHz – 3.0 GHz, SNR 25 dB.

Figure 25 - Numerical simulation. Actual (solid line) and reconstructed (dashed line) range profiles for an 16-cm-thick wall divided into 12 layers. Frequency range 0.8 GHz – 3.0 GHz, SNR 10 dB.

A case where the simulation and the reconstruction grids are apparently different from each other is shown in Figure 26. The two discontinuities
Experimental validation

have been recovered very well, within the resolution allowed by the reconstruction grid, even with the very pronounced differences in contrast that characterize the object function.

Figure 26 - Numerical simulation. Actual (solid line) and reconstructed (dashed line) range profiles for a 24-cm-thick wall divided into 12 layers. Frequency range 0.8 GHz – 3.0 GHz, SNR 25 dB. The discontinuities in the actual profile do not correspond to any interface between adjacent cells in the reconstruction lattice.

Finally, we tested our algorithm with the real data shown in Figure 27: the test object is a 3 cm thick plexiglas-styrofoam-plexiglas sandwich, and the frequency band is 1.7-2.6 GHz. The reconstructions obtained on a 5-layer and a 20-layer grid are shown in Figure 25 and Figure 26, respectively.
Figure 27 - Reconstruction from real measurements. Original (solid line) and reconstructed (dashed line) range profiles for the plexiglas-styrofoam-plexiglas sandwich. Reconstruction grid composed of 5 layers. Frequency range 1.7 GHz - 2.6 GHz.

Figure 28 - Reconstruction from real measurements. Original (solid line) and reconstructed (dashed line) range profiles for the plexiglas-styrofoam-plexiglas sandwich. Reconstruction grid composed of 20 layers. Frequency range 1.7 GHz - 2.6 GHz.
Experimental validation

The accuracy of the reconstruction is good in both cases, compatibly with the capability of locating the discontinuities offered by the reconstruction lattice.

For the simulations shown here, we observed computing times from about 30 minutes to two hours on a Pentium 4 processor using a non optimized computer code.
4.1 INTRODUCTION

In this chapter we focus on the ongoing research about the improvement of the method previously described. In fact, although the presented results are encouraging and prove the reliability of our procedure, nonetheless we have considered some aspects that need to be more carefully addressed.

Undoubtedly, the first effort which is necessary to enhance the entire process is to collect more data for the inversion procedure. Even if the measurement of the reflection coefficient (magnitude and phase) allow to gain a meaningful information about the layered structure at each frequency, the possibility to increase the total amount of knowledge by employing the same number of frequencies, or even less, can be appealing in order to speed up the permittivity profile reconstruction. This can be achieved, for example, by using a multi-sensor probe, such as an array of antennas composed by two or more elements. Moreover, the need to employ compact and reliable probes at the lowest possible cost is another constraint that has to be kept in mind not only during the design procedure of the probe but also in the development of a new methodology for the inversion procedure. It is quite easy to realize that an horn antenna
Introduction

does not satisfy these requirements and that other kind of sources does not allow to use the forward solver we have described since they are unable to perform the requested measurements. Therefore, as an improvement of the actual method, we are developing a new forward solver which will be able to treat the problem of an array of antennas placed in proximity of the multilayered structure. The aim is to recover the scattering parameters for each element of this array and to use these data for the inversion procedure.

In this case, we base our solver on a Method of Moments (MoM) which includes the Green’s function for multilayered medium. In the open literature there are several formulations which describe the different approaches to the problem [54]-[59]. On this basis we have decided to implement the formulation proposed by K. A. Michalski and R. Mosig in [60] which is based on Mixed-Potential Integral Equations (MPIEs). They provide the representation for both the electrical and magnetic dyadic Green’s function (DGF’s) for multilayered, plane-stratified, uniaxial medium. In order to obtain each component of the dyad, they introduce an equivalent transmission line along the axis normal to the stratification and they develop the mixed integral equations within this framework. The dyadic Green’s function in the spectral domain is expressed in terms of equivalent currents and equivalent voltages which excite the defined equivalent network. The space counterpart of each component is then found in closed form by applying the Sommerfeld identity. As a result, the mixed-potential representation of the electric field is given and can be used in the determination of the elements of the generalized impedance matrix.

In the following sections we will describe in detail the adopted procedure, giving particular emphasis to the MPIE formulation as well as
to the numerical and theoretical aspects regarding the approximation of the spectral functions obtained by using the Generalized Pencil of Function (GPOF) method.
4.2 MULTILAYERED GREEN’S FUNCTION

The aim of this section is to describe the guidelines of the implementation of the Green’s function for multilayered media in a Method of Moments code. In particular, the procedures to calculate the elements of the generalized impedance matrix are reported as well as the input parameters and formulas which are necessary for the evaluation of each Green’s function component. The first step is to correctly formulate the integral equation for the situation presented in Fig. 1.1(a), where known electrical and magnetic currents \( \mathbf{J}^i, \mathbf{M}^i \) are radiating an arbitrary shaped object embedded in a multilayered medium. An external equivalent problem is shown in Figure 29, where the same impressed currents \( \mathbf{J}^i, \mathbf{M}^i \), together with surface currents \( \mathbf{J}_s, \mathbf{M}_s \) produce the correct fields \( \mathbf{E}, \mathbf{H} \) exterior to \( S \) and null fields inside \( S \).

\[\text{Figure 29 - Object in a multilayered medium excited by known electric and magnetic currents. (a) Original physical configuration; (b) External equivalent problem.}\]
Multilayered Greens’ function

Since \((E, H) = (E^i + E^s, H^i + H^s)\), where \((E^i, H^i)\) are the impressed fields due to \((J^i, M^i)\) and \((E^s, H^s)\) are the scattered fields due to \((J_s, M_s)\). The boundary conditions at \(S\) impose that:

\[
M_s = -n \times \left( E^i + E^s [J_s, M_s] \right)_{S^+} \tag{4.1}
\]

\[
J_s = n \times \left( H^i + H^s [J_s, M_s] \right)_{S^+} \tag{4.2}
\]

where \(n\) is the outward unit vector normal to \(S\) and the subscript \(S^+\) indicates that the fields are evaluated as the observation point approaches \(S\) from the exterior region. If the media is linear, we can express the fields produced by an arbitrary current distribution \((J, M)\) as

\[
E = \left\langle G^{EJ}; J \right\rangle + \left\langle G^{EM}; M \right\rangle \tag{4.3}
\]

\[
H = \left\langle G^{HJ}; J \right\rangle + \left\langle G^{HM}; M \right\rangle \tag{4.4}
\]

where \(G^{PQ}(r|r')\) is the DGF relating \(P\)-type fields at \(r\) to \(Q\)-type currents \(r'\). Since the DGF’s for the layered medium of Fig.1(b) are available, one can use (4.3) and (4.4) to compute the impressed fields and to express the scattered fields in (4.1) and (4.2) in terms of the unknown currents \((J_s, M_s)\). In view of the hypersingular behavior of \(G^{EJ}\) and \(G^{HM}\), it is preferable to convert (4.3) and (4.4) into their mixed-potential forms before they are used in (4.1) and (4.2). However, since in layered media the scalar potential kernels associated with the horizontal and vertical
current components are in general different, in the method presented in [60] the scalar potential kernel have been modified for arbitrary current distributions and the final mixed-potential forms are

\[
E = -j\omega\mu_0 \left( \mathbf{G}^A \cdot \mathbf{J} \right) + \frac{1}{j\omega\varepsilon_0} \nabla \left( \left( \mathbf{K}^\phi \cdot \nabla \mathbf{J} \right) + \left( \mathbf{C}^\phi \cdot \mathbf{z} \right) \right) + \left( \mathbf{G}^{EM} \cdot \mathbf{M} \right) \quad (4.5)
\]

\[
H = \left( \mathbf{G}^{HJ} \cdot \mathbf{J} \right) - j\omega\varepsilon_0 \left( \mathbf{G}^F \cdot \mathbf{M} \right) + \frac{1}{j\omega\mu_0} \nabla \left( \left( \mathbf{K}^\phi \cdot \nabla \mathbf{M} \right) + \left( \mathbf{C}^\psi \cdot \mathbf{z} \right) \right) \quad (4.6)
\]

where the prime over the operator nabla indicates that the derivatives are with respect to the source coordinates. In addition, \( \mathbf{G}^A \) and \( \mathbf{G}^F \) are the DGF’s for the magnetic and electric vector potentials, respectively, \( \mathbf{K}^\phi \) and \( \mathbf{K}^\psi \) are the corresponding scalar kernels, and \( \mathbf{C}^\phi \) and \( \mathbf{C}^\psi \) are the correction factor which are associated with the longitudinal electric and magnetic currents, respectively. It is worth of notice that the two divergences are proportional to, respectively, the electric and magnetic charge densities. By using the mixed-potential representation given by the (4.5) and (4.6) in (4.1) and (4.2) we are able to express the scattered fields radiated by \( (\mathbf{J}_s, \mathbf{M}_s) \) in terms of the MPIE’s. Since the only difference with respect to the free space case is the dyadic nature of the vector potential kernels, these MPIE’s can be employed in the same numerical solution procedures developed for the free space case.

spectral domain green’s function

Let us examine a uniaxially anisotropic medium which is transversely unbounded with respect to the \( z \) axis and characterized by a \( z \)-dependent complex-valued permeability and permittivity dyadics
Multilayered Greens’ function

\[ \mu = l_x \mu_t + \hat{z} \hat{z} \mu_z \quad \text{and} \quad \varepsilon = l_x \varepsilon_t + \hat{z} \hat{z} \varepsilon_z, \]

respectively, where \( l_x \) is the transverse unit dyadic \( \hat{x} \hat{x} + \hat{y} \hat{y} \). We want to calculate the fields \((E, H)\) at an arbitrary point \( r \) due to a specific current distribution \((J, M)\) as shown in Figure 30a.

The fields are related to these source by the well known Maxwell’s equations

\[
\nabla \times E = -j \omega \mu_0 \mu \cdot H - M \\
\nabla \times H = j \omega \varepsilon_0 \varepsilon \cdot E + J.
\]

(4.7)

Since the medium is homogenous and of infinite extent in any \( z \)-transverse plane, the analysis can be simplified by the Fourier
Multilayered Greens’ function

transformation of all fields with respect to the transverse coordinates. Therefore we express any scalar field component as \( f(\rho) = f(\rho, z) \), where \( \rho = \hat{x}x + \hat{y}y \) is the projection of \( \rho \) on the (x,y) plane. The Fourier transform pair are reported hereafter:

\[
F\left[ f(\rho) \right] = \tilde{f}(k_\rho; z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\rho) \exp(jk_\rho \cdot \rho) \, d\rho \, dz \quad (4.8)
\]

\[
F^{-1}\left[ f(k_\rho; z) \right] = f(\rho) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k_\rho; z) \exp(-jk_\rho \cdot \rho) \, dk_x \, dk_y \quad (4.9)
\]

where \( k_\rho = \hat{x}k_x + \hat{y}k_y \). Upon applying (4.2) to (4.1) and separating the transverse and longitudinal parts of the resulting equation, we can further facilitate the analysis if a proper spectrum-domain coordinate system is defined. The new coordinate system is based on a rotation of the previous one \( (k_\rho, \hat{z} \times k_\rho) \) (see Figure 31) with unit vectors \( (\hat{u}, \hat{v}) \) expressed by:

\[
\hat{u} = \frac{k_x}{k_\rho} \hat{x} + \frac{k_y}{k_\rho} \hat{y} \\
\hat{v} = -\frac{k_y}{k_\rho} \hat{x} + \frac{k_x}{k_\rho} \hat{y} \quad (4.10)
\]

where \( k_\rho = \sqrt{k_x^2 + k_y^2} \).

If we express the transverse electric and magnetic fields as
Multilayered Greens’ function

\[
\tilde{E}_t = \hat{u}V^e + \hat{v}V^h \\
\tilde{H}_t \times \hat{z} = \hat{u}I^e + \hat{v}I^h
\]

(4.11)

Figure 31 - New rotated spectrum-domain reference system.

and project them on the new reference system we found the following relationships:

\[
\frac{dV^p}{dz} = -jk_z^p Z^p I^p + v^p \\
\frac{dI^p}{dz} = -jk_z^p Y^p V^p + i^p
\]

(4.12)

where the superscript \( p \) assumes the values of \( e \) or \( h \). Therefore, the components of \( \tilde{E}_t \) and \( \tilde{H}_t \) in the \((u,v)\) plane may be interpreted as voltages and currents on a transmission-line analogue of the medium along the \( z \) axis (see Figure 30(b)). Hence, the original vector problem has been reduced to a scalar one. It is important to underline that there are two transmission lines which are involved and associated, respectively, with the TM (\( p \) represents \( e \)) and TE (\( p \) is equal to \( h \)) partial fields. The
Multilayered Greens’ function

propagation wavenumbers and characteristic impedances and admittances of the transmission lines are:

\[ k_z^p = \sqrt{k_0^2 \varepsilon_r \mu_r - k_p^2}, \quad k_0 = \omega \sqrt{\varepsilon_0 \mu_0} \]

\[ Z^e = \frac{1}{Y^e} = \frac{k_z^e}{\omega \varepsilon_0 \varepsilon_r} \]

\[ Z^h = \frac{1}{Y^h} = \frac{\omega \mu_0 \mu_r}{k_z^h} \]  

(4.13)
equivalent transmission line

Let us investigate the solutions of the transmission line equations (4.12) for unit-strength impulsive source. Therefore, we indicate with \( V_i^p (z | z') \) and \( I_i^p (z | z') \) the voltage and current, respectively, at \( z \) due to a 1-A shunt current source at \( z' \), and let \( V_v^p (z | z') \) and \( I_v^p (z | z') \) the voltage and current, respectively, at \( z \) due to a 1-V series voltage source at \( z' \) (as shown in Figure 32).

\[ \begin{align*}
\text{(a)} & \\
& \begin{array}{c}
Z^p \\
\downarrow \quad k_z^p \quad \uparrow \\
\quad + \\
\quad 1\text{A} \\
\quad z' \\
\quad \downarrow \\
\quad z \\
\end{array} \quad V_i^p (z | z') \quad \begin{array}{c}
\quad + \\
\quad 1\text{V} \\
\quad \downarrow \\
\quad z' \\
\quad \downarrow \\
\quad z \\
\end{array} \quad I_v^p (z | z') \\
\end{align*} \]

\[ \text{(b)} \]

**Figure 32** - Network problems for the determination of the transmission line Green’s function.
Then, the DGF’s are formulated in terms of transmission line Green’s function (TLGF’s) and from (4.12) follows that:

\[
\frac{dV_i^p}{dz} = -jk_z^p Z^p I_i^p
\]

\[
\frac{dI_i^p}{dz} = -jk_z^p Y^p V_i^p + \delta(z - z')
\]

\[
\frac{dV_v^p}{dz} = -jk_z^p Z^p I_v^p + \delta(z - z')
\]

\[
\frac{dI_v^p}{dz} = -jk_z^p Y^p V_v^p
\]

These quantities possess useful reciprocity properties

\[
V_i^p (z | z') = V_i^p (z' | z)
\]

\[
I_v^p (z | z') = I_v^p (z' | z)
\]

\[
V_v^p (z | z') = -I_i^p (z' | z)
\]

\[
I_i^p (z | z') = -V_v^p (z' | z).
\]

The linearity of the transmission line equations (4.12) allows one to obtain \((V^p, I^p)\) at any point \(z\) by using the superposition integrals

\[
V^p = \left< V_i^p, i^p \right> + \left< V_v^p, v^p \right>
\]

\[
I^p = \left< I_i^p, i^p \right> + \left< I_v^p, v^p \right>.
\]

Let us consider a multilayered medium with piecewise constant parameters. We indicate the parameters related to layer \(n\) with boundaries
Multilayered Greens’ function

$z_n$ and $z_{n+1}$ by a subscript $n$. The transmission line analogue of the layered medium consists of a cascade connection of uniform transmission line sections, where section $n$ with terminals at $z_n$ and $z_{n+1}$ has propagation constant $k^p_{zn}$ and characteristic impedance $Z^p_n$. As previously said, to find the TLGF’s, we excite the transmission line network by unit-strength voltage and current sources at $z'$ in section $n$ and compute the voltage and current at $z$ in section $m$. The primed media parameters assume the values pertaining to layer $n$, while the unprimed ones are those of layer $m$. As illustrated in Figure 33, where the source section is investigated, $\Gamma^p_n$ and $\bar{\Gamma}^p_n$ are the voltage reflection coefficients looking to the left and right, respectively, out of the terminals of section $n$.

Figure 33 - Generalized reflection coefficient out of the terminals of section $n$ and for voltage and current point sources in a transmission-line section.

These coefficients are referred to $z_n$ and $z_{n+1}$ can be calculated by using the following relations

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Multilayered Greens’ function

\[ \tilde{\Gamma}_{n+1}^p = \frac{\Gamma_{n,n+1}^p + \tilde{\Gamma}_{n+1,n}^p}{1 + \Gamma_{n,n+1}^p \tilde{\Gamma}_{n+1,n}^p} \] (4.18)

\[ \tilde{\Gamma}_{n-1}^p = \frac{\Gamma_{n,n-1}^p + \tilde{\Gamma}_{n-1,n}^p}{1 + \Gamma_{n,n-1}^p \tilde{\Gamma}_{n-1,n}^p} \] (4.19)

where

\[ \Gamma_{ij}^p = \frac{Z_i^p - Z_j^p}{Z_i^p + Z_j^p} \] (4.20)

And \( t_n^p = e^{-j\frac{k_p}{\epsilon}d_n} \) with \( d_n = z_{n+1} - z_n \). These formulas come from the source-free transmission line equations (4.12) and the continuity of the voltages and currents at the line junctions. In order to obtain the reflection coefficient in the source layer \( n \) it is necessary to apply recursively (4.18) and (4.19) beginning at, respectively, the left and the right ends of the transmission line network.

If we consider the case \( m = n \), when \( z \) is within the source section the TLGF’s can be determined from (4.14) and (4.15) by using the properties (4.16)

\[
V_i^p(z'|z) = \frac{Z^p}{2} \left[ e^{-j k_{m,n}^p |z-z'|} + A \right]
\]

\[
A = \frac{1}{D^p_n} \left( R_{n1}^p e^{-j k_{m,n}^p (2z_{n+1}-(z+z'))} + R_{n2}^p e^{-j k_{m,n}^p ((z+z')-2z_n)} +
\right.
\]
\[
+ R_{n3}^p e^{-j k_{m,n}^p (2d_n+(z-z'))} + R_{n3}^p e^{-j k_{m,n}^p (2d_n-(z-z'))} \right)
\] (4.21)
Multilayered Greens’ function

where

\[ D_n^p = 1 - \Gamma_n^p \tilde{\Gamma}_n^p t_n^p \] (4.22)

\[ R_{n1}^p = \Gamma_n^p \quad ; \]
\[ R_{n2}^p = \tilde{\Gamma}_n^p \quad ; \]
\[ R_{n3}^p = R_{n4}^p = \Gamma_n^p \tilde{\Gamma}_n^p \]

The first term in (4.21) represents the direct ray between the source and the field point, while the second term represents the rays that undergo partial reflections at the upper and lower slab boundaries before reaching the observation point. All the others TLGF’s are presented hereafter:

\[ I_1^p (z | z') = \frac{1}{2} \left[ \text{sgn} (z - z') e^{-jk_n^p|z - z'|} + C \right] \]

\[ C = \frac{1}{D_n^p} \left( -R_{n1}^p e^{-jk_n^p(2z_{n+1} - (z + z'))} + R_{n2}^p e^{-jk_n^p((z + z') - 2z_a)} + R_{n3}^p e^{-jk_n^p(2d_a + (z - z'))} + R_{n3}^p e^{-jk_n^p(2d_a - (z - z'))} \right) \] (4.24)

\[ V_1^p (z | z') = \frac{1}{2} \left[ \text{sgn} (z - z') e^{-jk_n^p|z - z'|} + F \right] \]

\[ F = \frac{1}{D_n^p} \left( R_{n1}^p e^{-jk_n^p(2z_{n+1} - (z + z'))} - R_{n2}^p e^{-jk_n^p((z + z') - 2z_a)} - R_{n3}^p e^{-jk_n^p(2d_a - (z - z'))} + R_{n3}^p e^{-jk_n^p(2d_a + (z - z'))} \right) \] (4.25)
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\[
I_p^p (z \mid z') = \frac{Y_p^n}{2} \left[ e^{-j k_m^p |z-z'|} + G \right]
\]

\[
G = \frac{1}{D_n^p} \left( -R_{n1}^p e^{-j k_m^p (2z_{n1} - (z + z'))} - R_{n2}^p e^{-j k_m^p ((z + z') - 2z_n)} + \right.
\]

\[
+ R_{n3}^p e^{-j k_m^p (2d_n - (z - z'))} + R_{n3}^p e^{-j k_m^p (2d_n + (z - z'))} \right)
\]

(4.26)

In the case that \( m \) is not equal to \( n \), this means that \( z \) is outside the source section and \( z < z' \). Given the voltage \( V_p(z_n) \) across the left terminals of section \( n \), the voltage \( V_p(z) \) and current \( I_p(z) \) at any point \( z \) in section \( m \) can be found from the homogeneous form of the transmission-line equations (4.12) (see Appendix A).

The next step is to evaluate the dyadic Green’s function in spectral domain by using the formulation obtained before. If we consider only electrical currents we can express the fields in terms of

\[
\mu_0 \mu \cdot H = \nabla \times A
\]

\[
E = -j \omega A - \nabla \Phi
\]

(4.27)

and the linearity of the problem allow us to write

\[
A = \mu_0 \left\{ G^A : J \right\}
\]

(4.28)

where \( G^A(r \mid r') \) is the vector potential DGF. From (4.4) and (4.27) it follows that that

\[
\mu G^{HJ} = \nabla \times G^A.
\]

(4.29)
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The derivations are simplified in the spectral domain and, even if the (4.29) does not uniquely specify $G^A$, the from postulated in the code (which follows [60]) is

$$
\widetilde{G}^A = I \tilde{G}_{vv}^A + \tilde{z}\hat{u}\tilde{G}_{zu}^A + \tilde{z}\hat{z}\tilde{G}_{zz}^A
$$

(4.30)

which is consistent with the Sommerfeld’s choice of potentials for a horizontal Hertzian dipole over a dielectric half-space. In matrix form we have

$$
\begin{bmatrix}
\tilde{G}^A
\end{bmatrix} =
\begin{bmatrix}
\tilde{G}_{vv}^A & 0 & 0 \\
0 & \tilde{G}_{vv}^A & 0 \\
\frac{k_x}{k_x} \tilde{G}_{zu}^A & \frac{k_y}{k_y} \tilde{G}_{zu}^A & \tilde{G}_{vv}^A
\end{bmatrix}
\begin{bmatrix}
\tilde{G}_{vv}^A & 0 & 0 \\
0 & \tilde{G}_{vv}^A & 0 \\
\frac{k_x}{k_x} \tilde{G}_{zu}^A & \frac{k_y}{k_y} \tilde{G}_{zu}^A & \tilde{G}_{vv}^A
\end{bmatrix}
\begin{bmatrix}
\tilde{G}^A
\end{bmatrix}
$$

(4.31)

which indicates that horizontal and vertical components of the vector potential are involved for a horizontal current source. All the components are expressed in terms of the TLGF’s in the following

$$
\tilde{G}_{vv}^A(k_\rho; z | z') = \frac{1}{j\omega\mu_0} V^h_i
$$

(4.32)

$$
\tilde{G}_{zz}^A(k_\rho; z | z') = \eta_0^2 \frac{\mu_1}{j\omega\mu_0\varepsilon_z} I^e_v
$$

(4.33)

$$
\tilde{G}_{zu}^A(k_\rho; z | z') = \frac{\mu_1(I^h_i - I^e_i)}{jk_\rho}
$$

(4.34)
The scalar potential kernel $K^\Phi$ and the correction factor $C^\Psi$ are defined by

$$\tilde{K}^\Phi(k_\rho; z | z') = -\frac{j \omega e_0}{k_\rho^2} \left( V^h_i - V^e_i \right)$$

(4.35)

$$\tilde{C}^\Phi(k_\rho; z | z') = \frac{\omega^2 \varepsilon_0 \mu_0 \mu'_i}{k_\rho^2} \left( V^h_v - V^e_v \right)$$

(4.36)

The space-domain counterparts of the spectral kernels previously derived in this section can be expressed in terms of the Sommerfeld integrals by using the following relations

$$F^{-1} \left[ \begin{array}{c} \sin(n \zeta) \\ \cos(n \zeta) \end{array} \right] \tilde{f}(k_\rho) = (-j)^n \left[ \begin{array}{c} \sin(n \varphi) \\ \cos(n \varphi) \end{array} \right] S_n \{ \tilde{f}(k_\rho) \}$$

$$n = 0, 1, 2$$

(4.37)

where

$$S_n \{ \tilde{f}(k_\rho) \} = \frac{1}{2\pi} \int_0^\infty \tilde{f}(k_\rho) J_n(k_\rho \rho) k_\rho dk_\rho$$

(4.38)

is the Sommerfeld integral. The term $J_n$ is the Bessel function of order $n$ and $(\rho, \varphi)$ are the cylindrical coordinates of the projection of the field point on the $(x, y)$ plane. If the source is off the $z$ axis, the generalization for arbitrary source location imposes the following substitutions
Multilayered Greens’ function

\[ \rho \rightarrow \zeta = \left| \rho - \rho' \right| \]
\[ \varphi \rightarrow \phi = \arctan \frac{y - y'}{x - x'} \] \hspace{1cm} (4.39)

Therefore the space-domain counterparts of the derived spectral kernels can be expressed in term of the Sommerfeld integrals by using (4.37) and (4.38). Hence, it is possible to evaluate:

\[ E = -j \omega \mu_0 \left< G^A; J \right> + \frac{1}{j \omega \varepsilon_0} \nabla \left< \left( \nabla^\phi, \nabla' \cdot J \right) + \left( C^\phi \hat{z}; J \right) \right> \] \hspace{1cm} (4.40)

As an example we show two components hereafter:

\[ G^A_{xx}(\rho; z \mid z') = G^A_{yy}(\rho; z \mid z') = S_0 \left\{ \tilde{G}^A_{vv}(k_\rho; z \mid z') \right\} = \]
\[ = \frac{1}{2\pi} \int_0^\infty \tilde{G}^A_{vv}(k_\rho; z \mid z') J_0(k_\rho \rho) k_\rho dk_\rho \] \hspace{1cm} (4.41)

and the final result produces

\[ G^A = \begin{pmatrix}
G^A_{xx} & 0 & 0 \\
0 & G^A_{yy} & 0 \\
G^A_{zx} & G^A_{zy} & G^A_{zz}
\end{pmatrix} \] \hspace{1cm} (4.42)
4.3 SPACE DOMAIN GREEN’S FUNCTIONS

The evaluation of Sommerfeld integrals is a critical factor for the efficient and accurate MoM analysis since the analytical solution of the Sommerfeld integral is not available and the numerical integration is time consuming since the integrand is both highly oscillating and slowly decaying. A brief overview of the topology of the addressed problem in the complex plane can help to better realize the intrinsic difficulties of a numerical integration [60]. Looking at the geometry presented in Figure 34, we note that, since there is an infinite dielectric layer of air, there is a branch-point singularity in the spectral domain expression of the Green’s functions, specifically at \( k_\rho = k_0 \).

![Figure 34 – Typical layered medium and its relative topology in the complex plane.](image)

In addition to the branch-point singularities, there are some surface waves poles between the minimum and the maximum wavenumbers involved in the geometry. Their number is dependent on the electrical thickness and dielectric constant of the layers involved. It is therefore necessary to deform the pattern of integration since the presence of these singularities do not allow to stay along the real axis for the entire calculation.
example of a possible numerical approach for the integration is illustrated in Figure 35.

**Figure 35** – Deformation of the Sommerfeld integration path. The new path (red line) allows to avoid integration in proximity of the poles which would produce numerical errors.

In order to prevent an integration path too close to the poles, which can cause high numerical instability, it is preferable to define an integration path which bypasses the singularities and then returns on the real axis. Therefore, as an example, the integration in (4.41) is performed along the intervals (0,A), (A,B) and (B,C) by using standard integration rules and thus only remains the term:

\[
\frac{1}{2\pi} \int_{k_0}^{k_{\text{max}}} \tilde{G}_{vv}^A(k_\rho, z | z') J_0(k_\rho \rho) k_\rho \, dk_\rho .
\]  

(4.43)

At this point we can still recur to numerical integration even if we have to use proper techniques, such as weighting averages or continued fraction expansion [61]-[65], in order to face the oscillating behavior of the Bessel function, which causes serious problem to the integration convergence. On the contrary, we can also exploit the asymptotic behavior of the function in the spectral domain and thus regularizing the integrand:
Space domain Greens’ functions

\[
\lim_{k_{p} \to \infty} \tilde{G}^{A}_{v;v}(k_{p};z \mid z') k_{p} = G_{\infty},
\]

\[
\frac{1}{2\pi} \int_{0}^{\infty} \left[ \tilde{G}^{A}_{v;v}(k_{p};z \mid z') k_{p} - G_{\infty} \right] J_{0}(k_{p}\rho) dk_{p} + \frac{G_{\infty}}{2\pi} \int_{0}^{\infty} J_{0}(k_{p}\rho) dk_{p}.
\]

However, many recent efforts to further accelerate the computation have been presented [66],[67] and among them the discrete complex image method (DCIM) [68]-[73] is particularly appealing and it has been implemented in our code. This approach was first proposed for a thick single layer microstrip geometry, and later extended to multilayer planar geometries with arbitrary thickness. It takes advantages of the fact that the Sommerfeld integral can be integrated analytically if the integrand, apart from the kernel \(J_{0}(k_{p}\rho)\), can be approximated with complex exponentials. With this approach the numerical problem concerning the integration techniques mentioned above is eliminated. Physically, each term of the series represents a discrete image with complex amplitude and location (see Figure 36).

**Figure 36** – The problem of a real source immersed in a layered medium is transformed, through the application of the DCIM, in an equivalent problem with the real source together with its complex images. Each source contributes to the field at the observation point (continuous black and dashed grey lines).
However, in this case, the burden of the numerical algorithm falls on the task of approximating the spectral-domain Green’s functions in terms of complex exponentials. An exponential approximation technique which has proved to be reliable and fast is the so called Generalized Pencil of Function (GPOF) method [74] which has been employed in casting the Green’s functions into closed forms [75][74]. For further details on its implementation see Appendix C.

In order to calculate the space domain Green’s functions we have implemented the procedure illustrated in [76] that is here reported. In this method, a two-level approach is adopted since the range of approximation is divided into two parts: the former covers the region where the function to be approximated has rapid transitions while the latter is applied in the second region where the function is smooth. The situation is reported in Figure 37.

![Figure 37](image)

**Figure 37** – Definition of the integration path for the two-level approximation.

If we reconsider the (4.43), our aim is to approximate the spectral component of the Green’s function in terms of complex exponential to perform an analytical evaluation of the integral. This is possible via the Sommerfeld identity:
It is clear that the crucial step in the derivation of the closed-form Green’s functions is the exponential approximation of the spectral components. Since the GPOF method requires uniform samples along a real variable of a complex-valued function, the integration path cannot be chosen along the real $k_\rho$ axis. In fact, this operation produces an approximation in terms of exponentials of $k_\rho$ which cannot be cast into a form of exponentials of $k_z$ as required in the application of (4.45). Hence we need to deform the integration path. The two-level approach exploited starts with the approximation along the path $C_{ap1}$ (see Figure 37). The first step is to map a real variable $t$ onto the complex $k_z$ plane by using the relation:

$$k_{zi} = - j k_i [T_{o2} + t] \quad 0 \leq t \leq T_{o1}. \quad (4.46)$$

where $k_i$ is the wavenumber of the dielectric layer which hosts the source. The value of $T_{o2}$ is such that $k_{\rho max2} > k_m$ where $k_m$ is the maximum value of the wavenumber involved in the geometry and

$$k_{\rho max2} = k_i \sqrt{1 + T_{o2}^2}. \quad (4.47)$$

Since

$$k_\rho^2 = k_i^2 - k_{zi}^2 \quad (4.48)$$
the values of \( k_{zi} \) in (4.46) are within the interval \([-j k_{i} T_{o2}, -j k_{i}(T_{o2} + T_{o1})]\) while the \( k_{\rho} \) values span precisely from \( k_{\rho \text{max}2} \) to \( k_{\rho \text{max}1} \). which is clearly equal to:

\[
k_{\rho \text{max}1} = k_{i} \sqrt{(T_{o2} + T_{o1})^2}.
\]  

(4.49)

The choice of \( T_{o1} \) is not very critical as long as one choose \( k_{\rho \text{max}1} \) large enough to pick up the behavior of the spectral-domain Green’s function for large \( k_{\rho} \) values. The next step is to sample the function along the path \( C_{ap1} \) approximating it by using the GPOF method. The sampling operation is done just uniformly varying \( t \) between 0 and \( T_{o1} \). It is not necessary to use a large number of samples since in the interval \([k_{\rho \text{max}2}, k_{\rho \text{max}1}]\) the Green’s functions are always quite smooth. The result is an exponential approximation \( f(k_{\rho}) \) of the spectral Green’s function within the considered interval:

\[
f(k_{\rho}) = \sum_{n=1}^{N_{1}} b_{in} e^{\beta_{i} n t} = \sum_{n=1}^{N_{1}} a_{in} e^{-\alpha_{i} n k_{zi}}
\]  

(4.50)

where

\[
\alpha_{in} = \frac{\beta_{in}}{j k_{i}} ; \quad a_{in} = b_{in} e^{-j k_{i} \alpha_{in} T_{o2}}.
\]  

(4.51)

All the coefficients and exponents are obtained from the GPOF method and \( N_{1} \) is the number of exponentials is based upon the number of significant singular values obtained in this approximation. The following
step consists in subtracting the function approximated within the range
\([k_{\rho_{\text{max}2}}, k_{\rho_{\text{max}1}}]\) from the original function to be approximated in order to
have non-zero values of the remaining function over the range \([0, k_{\rho_{\text{max}2}}]\).
The remaining function is then sampled along the path \(C_{\text{ap}2}\) and the GPOF
method is applied again so that, at the end of the entire described
procedure, we have the following approximation of the spectral Green’s
function

\[
\hat{G}^A_{vv}(k_{\rho} ; z | z') \equiv \frac{1}{j 2 k_{zi}} \left[ \sum_{n=1}^{N_1} a_{1n} e^{-\alpha_{1n} k_{zi}} + \sum_{n=1}^{N_1} a_{2n} e^{-\alpha_{2n} k_{zi}} \right] \quad (4.52)
\]

It is now straightforward to apply the (4.45) at each term to obtain the
spatial domain Green’s function

\[
G^A_{xx}(\rho) = \sum_{n=1}^{N_1} \frac{a_{1n}}{r_{1n}} e^{-j k_{\rho} r_{1n}} + \sum_{n=1}^{N_2} \frac{a_{2n}}{r_{2n}} e^{-j k_{\rho} r_{2n}} \quad (4.53)
\]

where

\[
r_{1n} = \sqrt{x^2 + y^2 - \alpha_{1n}^2}, \quad r_{2n} = \sqrt{x^2 + y^2 - \alpha_{2n}^2}.
\] (4.54)

If we now reconsider the general dyadic form of

\[
E = -j \omega \mu_0 \left\langle \mathcal{G}^A, \mathcal{J} \right\rangle + \frac{1}{j \omega \varepsilon_0} \nabla \left( \left\langle \mathcal{K}^\phi, \nabla \cdot \mathcal{J} \right\rangle + \left\langle C^\phi z, \mathcal{J} \right\rangle \right) \quad (4.55)
\]
we have all the elements to evaluate it (see also Appendix C). We postulate to expand the surface current densities in terms of a linear combination of the basis functions as follows

\[ J = \sum_n I_n f_n \quad (4.56) \]

and then to calculate the first term of (4.55) we perform

\[
\left\langle G^A, J \right\rangle = \int_S G^A \cdot J \, dS = \sum_i \int_S G_{ji} J_i \, dS = \int_S \left( G_{xx}^A \cdot J_x + G_{xy}^A \cdot J_y + G_{xz}^A \cdot J_z \right) \, dS = \int_S \left( G_{yx}^A \cdot J_x + G_{yy}^A \cdot J_y + G_{yz}^A \cdot J_z \right) \, dS = \int_S \left( G_{zx}^A \cdot J_x + G_{zy}^A \cdot J_y + G_{zz}^A \cdot J_z \right) \, dS, \quad (4.57)
\]

The dyad element \( G_{zx} \) is not directly evaluated by using the GPOF as well as the term \( G_{zy} \). Instead of the \( G_{zx} \) term the procedure produces \( \int G_{zx}^A \, dx \). Therefore we have to make a choice between two options. The first one is to derive with respect to \( x \) and then obtain \( G_{zx} \). The other one is to wait.
until we have to perform the weighting procedure and then, instead of
deriving \( \int G^A_{z \nu} dx \) we derive the current as shown

\[
\int_S \left( \frac{\partial}{\partial x} \int G^A_{z \nu} dx \right) J dS = \int_S \left( \int G^A_{z \nu} dx \right) \left( \frac{\partial}{\partial x} J \right) dS \tag{4.58}
\]

The second term in (4.55) involves the scalar potential kernel \( K^\Phi \) and the
correction factor \( C^\Psi \) and they may be evaluated by using:

\[
\left\langle C^\phi \hat{z}; J \right\rangle = \int_S C^\phi J \hat{z} dS, \tag{4.59}
\]

\[
\left\langle K^\Phi; (\nabla \cdot J) \right\rangle = \int_S K^\Phi (\nabla \cdot J) dS = \int_S K^\Phi \left( \nabla \cdot \left( \sum_n I_n f_n \right) \right) dS =
\]

\[
= \int_S K^\Phi \sum_n I_n (\nabla \cdot f_n) dS = \int_S K^\Phi \sum_n I_n b_n dS \tag{4.60}
\]

where \( b_n \) is the well known surface charge density associated with the
basis element. If we compare the multilayered case to the free space one
we finally obtain:

\[
E = -j \omega \mu_0 A + \frac{1}{j \omega \varepsilon_0} \nabla \Phi
\]

\[
E = -j \omega \mu_0 \left( \frac{G^A_{z \nu}; J}{A} \right) + \frac{1}{j \omega \varepsilon_0} \nabla \left( \left\langle K^\Phi, \nabla \cdot J \right\rangle + \left\langle C^\phi \hat{z}; J \right\rangle \right). \tag{4.61}
\]
Space domain Greens’ functions
CONCLUSIONS

In the present work we have presented a reliable and efficient algorithm for the data inversion, which is based on a fully nonlinear data model in conjunction with an optimization technique.

The reconstruction of the permittivity range profile has been tested both on synthetic and real data to validate the electromagnetic code as well as to assess the accuracy and efficiency of the reconstruction procedure. We have studied the resolution of the algorithm and its robustness to the noise, demonstrating the ability of our procedure to be able to recognize the presence of high discontinuities even independently from the discretization fixed by the user.

As a part of the ongoing improvement of the presented method, we have addressed the implementation of a new optimization algorithm, namely the Particle Swarm Optimization, which has been customized and enhanced for our purposes.

Finally, a detailed description of a fast and efficient procedure to evaluate the Green’s function for a multilayered medium has been given. This is the groundwork useful for the next step toward a more reliable and versatile forward solver to be implemented in the inversion procedure.
Conclusions
In the case that \( m < n \), this means that \( z \) is outside the source section and \( z < z' \). Given the voltage \( V^p(z_n) \) across the left terminals of section \( n \), the voltage \( V^p(z) \) and current \( I^p(z) \) at any point \( z \) in section \( m \) can be found from the homogeneous form of the transmission-line as showed in the following expressions.

\[
\begin{align*}
V^p_i(z | z') &= V^p_i(z_n | z') \frac{\prod_{k=m+1}^{n-1} \left( 1 + \frac{\bar{\Gamma}_k^p}{1 + \bar{\Gamma}_k^p e^{-j2k_m^p(z-z_n)}} \right)}{1 + \bar{\Gamma}_m^p e^{-j2k_m^p(z-z_n)}} \left[ 1 + \bar{\Gamma}_m^p e^{-j2k_m^p(z-z_n)} \right] e^{-jk_m^p(z_{m+1}-z)} \\
I^p_i(z | z') &= V^p_i(z_n | z') \frac{\prod_{k=m+1}^{n-1} \left( 1 + \frac{\bar{\Gamma}_k^p}{1 + \bar{\Gamma}_k^p e^{-j2k_m^p(z-z_n)}} \right)}{1 + \bar{\Gamma}_m^p e^{-j2k_m^p(z-z_n)}} \left[ 1 - \bar{\Gamma}_m^p e^{-j2k_m^p(z-z_n)} \right] e^{-jk_m^p(z_{m+1}-z)} \\
V^p_v(z | z') &= V^p_v(z_n | z') \frac{\prod_{k=m+1}^{n-1} \left( 1 + \frac{\bar{\Gamma}_k^p}{1 + \bar{\Gamma}_k^p e^{-j2k_m^p(z-z_n)}} \right)}{1 + \bar{\Gamma}_m^p e^{-j2k_m^p(z-z_n)}} \left[ 1 + \bar{\Gamma}_m^p e^{-j2k_m^p(z-z_n)} \right] e^{-jk_m^p(z_{m+1}-z)} \\
I^p_v(z | z') &= V^p_v(z_n | z') \frac{\prod_{k=m+1}^{n-1} \left( 1 + \frac{\bar{\Gamma}_k^p}{1 + \bar{\Gamma}_k^p e^{-j2k_m^p(z-z_n)}} \right)}{1 + \bar{\Gamma}_m^p e^{-j2k_m^p(z-z_n)}} \left[ 1 - \bar{\Gamma}_m^p e^{-j2k_m^p(z-z_n)} \right] e^{-jk_m^p(z_{m+1}-z)}
\end{align*}
\]
Appendix

The product in the formulas (which are applicable irrespective of the source type) is equal to one if the lower limit exceeds the upper limit. Analogous formulas can be developed for the case $m > n$

\[
V_i^p(z | z') = V_i^p(z_{n+1} | z') \frac{\prod_{k=n+1}^{m-1} \left(1 + \tilde{\Gamma}_k^p\right)e^{-jk_dk}}{1 + \tilde{\Gamma}_m^p e^{-j2k_dz_m}} \left[1 + \tilde{\Gamma}_m^p e^{-j2k_d(z_m-z)}\right] e^{-jk_d(z-z_m)}
\]

\[
I_i^p(z | z') = I_i^p(z_{n+1} | z') \frac{\prod_{k=n+1}^{m-1} \left(1 + \tilde{\Gamma}_k^p\right)e^{-jk_dk}}{1 + \tilde{\Gamma}_m^p e^{-j2k_dz_m}} \left[-Y_m^p\right] \left[1 + \tilde{\Gamma}_m^p e^{-j2k_d(z_m-z)}\right] e^{-jk_d(z-z_m)}
\]

\[
V_v^p(z | z') = V_v^p(z_{n+1} | z') \frac{\prod_{k=n+1}^{m-1} \left(1 + \tilde{\Gamma}_k^p\right)e^{-jk_dk}}{1 + \tilde{\Gamma}_m^p e^{-j2k_dz_m}} \left[1 + \tilde{\Gamma}_m^p e^{-j2k_d(z_m-z)}\right] e^{-jk_d(z-z_m)}
\]

\[
I_v^p(z | z') = I_v^p(z_{n+1} | z') \frac{\prod_{k=n+1}^{m-1} \left(1 + \tilde{\Gamma}_k^p\right)e^{-jk_dk}}{1 + \tilde{\Gamma}_m^p e^{-j2k_dz_m}} \left[-Y_m^p\right] \left[1 + \tilde{\Gamma}_m^p e^{-j2k_d(z_m-z)}\right] e^{-jk_d(z-z_m)}
\]
APPENDIX B

If $\mathbf{U}$ and $\mathbf{V}$ are two vectors as shown

$$\mathbf{U} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$$
$$\mathbf{V} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

their dyad product is

$$\mathbf{UV} = u_1 v_1 \hat{ii} + u_1 v_2 \hat{ij} + u_1 v_3 \hat{ik} + u_2 v_1 \hat{ji} + u_2 v_2 \hat{jj} + u_2 v_3 \hat{jk} + u_3 v_1 \hat{ki} + u_3 v_2 \hat{kj} + u_3 v_3 \hat{kk}$$

that can be rewritten as

$$\mathbf{UV} = \mu_{11} \hat{ii} + \mu_{12} \hat{ij} + \mu_{13} \hat{ik} + \mu_{21} \hat{ji} + \mu_{22} \hat{jj} + \mu_{23} \hat{jk} + \mu_{31} \hat{ki} + \mu_{32} \hat{kj} + \mu_{33} \hat{kk}$$

or that can be arranged in matrix form as

$$\mathbf{UV} = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix}.$$
Appendix

which means that the dyadic product is not commutative. Another proof of this is illustrated in the following

\[ \mathbf{V} \cdot \mathbf{M} = \sum_i v_i \mu_{ij}, \]

\[ \mathbf{M} \cdot \mathbf{V} = \sum_j v_j \mu_{ij}, \]

\[ \mathbf{V} \cdot \mathbf{M} \neq \mathbf{M} \cdot \mathbf{V}. \]
APPENDIX C

The computation of Sommerfeld integral tails using the Generalized Pencil of Function Method is here briefly presented. Suppose we want to solve the Sommerfeld-type integral:

\[
I = \int_{0}^{\infty} \tilde{G}(z, z'; \lambda) J_n(\lambda \rho) \lambda \, d\lambda
\]

where \( \tilde{G} \) is generic spectral domain Green’s function, \( J_n \) is the Bessel function of the first kind, \( \lambda \) is the radial wave number and \( \rho \) is the horizontal distance between the field and source points in the \( x \)-\( y \) plane while their vertical components are \( z \) and \( z' \).

Let us focus on the integrand of \( I \), hereafter referred as \( f(\lambda) \) on the path \([a, \infty)\). Suppose we have a total of \( N = mK \) samples and we define \( t = \lambda - a \). Through the GPOF we will be able to approximate \( f(t) \) on \([0, \infty)\) by a sum of \( M \) exponentials as

\[
f(t) = \sum_{i=1}^{M} R_i e^{s_i t}
\]

where \( R_i \) is called the residue and \( s_i \) is the exponent. If we sample \( f(t) \) with a sampling step \( \Delta T = q/m \) then we obtain:
Appendix

\[ f_p = f(p \Delta T) = \sum_{i=1}^{M} R_i \gamma_i^p \quad \text{for } p = 0,1,\ldots, N-1 \]

where the poles are \( \gamma_i = e^{s_i \Delta T} \) for \( i=1,2,\ldots,M \). In order to solve for the \( R_i \) and \( s_i \) using the samples \( f_p \), we have to construct the Hankel matrix as

\[
\begin{bmatrix}
  f_0 & f_1 & \cdots & f_L \\
  f_1 & f_2 & \cdots & f_{L+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{N-L-1} & f_{N-L} & \cdots & f_{N-1}
\end{bmatrix}
\]

and perform the singular value decomposition (SVD):

\[
Y = U \Sigma V^H
\]

where \( \Sigma \) is a diagonal matrix with the singular values \( \sigma_i \) of the \( Y \). The number \( M \) of exponentials can be chosen by analyzing the values of \( \sigma_i \) and discarding the values which are \( \sigma_i / \sigma_{\text{max}} < \text{tol} \), where \( \sigma_{\text{max}} \) is the largest singular value and \( \text{tol} \) is a set tolerance. Smaller values of \( \text{tol} \) means that a larger number of exponentials is used to approximate \( f(t) \) yielding to more accurate results but requiring a longer computational time. Once \( M \) is fixed, the first \( M \) columns of \( U \) are used to build a new \( MxM \) matrix using a least squares methods, and the eigenvalues of this matrix are the poles \( \gamma_i \). Finally, the solution of the leas squares problem
\[
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
\gamma_1 & \gamma_2 & \ldots & \gamma_M \\
\ldots & \ldots & \ldots & \ldots \\
\gamma_1^{N-1} & \gamma_2^{N-1} & \ldots & \gamma_M^{N-1}
\end{bmatrix}
\begin{bmatrix}
R_1 \\
R_2 \\
\ldots \\
R_M
\end{bmatrix}
= 
\begin{bmatrix}
f_1 \\
f_2 \\
\ldots \\
f_{N-1}
\end{bmatrix}
\]
gives the values of \( R_i \) while the values of the \( s_i \) can be obtained from:

\[
s_i = \frac{\log(\gamma_i)}{\Delta T}.
\]
REFERENCES


Appendix


Appendix


Appendix


Appendix


